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# THE MATHEMATICS TEACHER

## THE PSYCHOLOGY OF PROBLEM SOLVING

By EDWARD L. Thorndike
Institute of Educational Research
Teachers College, Columbia University

*(Continued)*

### THE OVERVALUATION OF VERBAL PROBLEMS

One reason for the great value attached to solving these verbal problems is a confusion of their value as training with their value as tests, and a misunderstanding of what they test. The ability to organize a set of facts in an equation or set of equations such that solving will produce the desired answer is very closely correlated with general intelligence of the scholarly type. The pupils who can do it well rank high in intellect and scholarship. So it is natural to infer that doing it creates and improves the ability. But this inference may be false or at least much exaggerated. Ability in supplying the missing words in sentences is also an excellent test of general intelligence. But the ability certainly has not been created or improved by supplying missing words, since that form of mental gymnastics has not been experienced by pupils save as a feature of psychological tests! The close correlation between ability in solving verbal problems and general ability is perhaps sufficiently accounted for by the fact that the task involves two abilities, each of which is closely related to general ability, namely ability in algebraic computation and ability in paragraph reading. Given a sufficient ability in algebraic computation and in paragraph reading, and pupils might conceivably solve a novel problem almost as well after two hours training in problem solving as after two hundred. Training of course improves their ability to solve the special sorts of problems they practice with, but the value of this depends largely on the genuineness and usefulness of the particular problems used.

Certain students of the teaching of algebra would agree with all this, but insist that the value of the verbal problems as train-

ing in the exact and adequate reading of paragraphs was sufficient to justify the high value attached to them.

This could conceivably be true. Solving a thousand verbal problems certainly has whatever educative value belongs to reading with great care a thousand short paragraphs and doing the thousand relevant computations. It has, indeed, the additional value that belongs to organizing the facts thus carefully read into equational forms such as will give the desired answers. The reading matter of these thousand short paragraphs is, however, so little in amount and so specialized in its nature that the training given by it seems insufficient to justify the high opinion of verbal problems or the time devoted to solving them.

### The Use of Problems to Show the Need for a Certain Procedure and to Aid in Mastering It as Well as to Test and Improve the Ability to Apply the Procedure

Other things being equal, it is better for pupils to feel some need for a procedure and purpose in learning it before they learn it. They are then more likely to understand it and much more likely to care about learning it.* Thus writing a real letter is now the beginning rather than the end of the lessons about "Dear Sir" and "Yours truly"; problems about the total cost of several toys or Christmas presents are the beginning rather than the end of the lessons on "carrying" in addition. "Why do we open the draughts of a stove to make the fire burn?" and, "What do we mix with the gasoline in an automobile?" are questions that introduce rather than follow the study of oxygen.

Thus in algebra problems about the average temperature of a series of days varying above and below 0, or about the total of certain credits and penalties in a rating may be excellent features in the introduction to the addition of negative numbers. Problems like the following may be useful as parts of an introduction to "— divided by — gives +."*

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*We do not here discuss this general educational axiom because probably it will be acceptable as stated. The whole matter of pupils' purposes in learning, and the special doctrine of "first the need, then the technique," has received a classic general treatment at the hands of Dewey. The case with algebra is much the same as with arithmetic, on which the reader may consult Chapter XIV of "The Psychology of Arithmetic" (Thorndike, '22).

Four boys are rated for strength in comparison with the average for their age.

|            |    |
|------------|----|
| Arthur is  | 20 |
| John is    | 12 |
| Fred is    | 4  |
| Bert is    | 8  |

Supply the missing numbers:

Arthur is........times as far below the average as John.
Arthur is........times as far below the average as Fred.
Arthur is........times as far below the average as Bert.
John is........times as far below the average as Fred.

Other things are not always equal. There may be no vital, engaging problems to use as introductory material. For examples, there is not, to my knowledge, any problem that is vital and engaging to the average high-school pupil by which to introduce the general symbolism of fractional exponents. Such problems, though in existence, may use up more time than can be spared. There is, for instance, a problem almost perfectly adapted to arouse the need for knowledge of the laws of signs in multiplication, namely, the problem of measuring resemblance between a pair of measures both of which are divergences from a type or average. But it takes so long to teach the meaning of "resemblance" in such cases that probably the game is not worth the candle. The procedure may be so intrinsically valuable and interesting that mere contact with it will quickly inspire a desirable purpose and activity. For example, gifted pupils will probably learn that $\sqrt{a}\ \sqrt{a} = a$ as readily by straightforward consideration of $(\sqrt{4}\ \sqrt{4})$, $(\sqrt{9}\ \sqrt{9})$, $(\sqrt{16}\ \sqrt{16})$, $(\sqrt{2}\ \sqrt{2})$, and $(\sqrt{3}\ \sqrt{3})$, as by any introductory problem to display the need of knowing that the square root of any number times the square root of the same number equals the number.

### CRITERIA IN SELECTING PROBLEMS

In this section, as in the previous one, we are concerned not alone with problems where an equation or equations are used to discover certain particular quantities relating to one particular state of affairs, but with problem material in general.

Solving problems in school is for the sake of problem solving in life. Other things being equal, problems where the situation is real are better than problems where it is described in words.

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*The illustrations here are not problems where organization in the equational form is necessary. What is said about problems in this section, indeed concerns all problems of types I and II, not merely the II-B-b problems.

Other things being equal, problems which might really occur in a sane and reasonable life are better than bogus problems and mere puzzles. Other things being equal, problems which give desirable training in framing equations from the realities or the verbal statements are better than problems which give training chiefly in solving the equations when framed. The latter training can be got easily by itself.

As was suggested in an earlier article, a better selection of problems will probably be secured if, instead of searching for problems which conveniently apply to fractional equations, problems to apply simultaneous linear equations, and so on, we search for problems which are intrinsically worth learning to solve by algebraic methods.

If it happens that there are no genuine, important problems calling for the framing and solution of a certain technique, say simultaneous quadratics, we may simply leave that technique without application to verbal problems or we may frankly provide problems that make no false pretenses at reality as in "I am thinking of two numbers, . . . etc."

This case of simultaneous quadratics is a good one to illustrate the two points of view contrasted here. The older view, in order to have applications of simultaneous quadratics, fabricated extraordinary tasks depending on insane curiosity to know the dimensions of a field which, when altered in various ways, gives fields of certain areas, and the like. The newer view selects first the case of determining the constants in a quadratic equation from knowledge of the $(x)$ $(y)$ values of certain points on the curve. The ability to do this is not of great "social utility" to many of the individuals who study ninth or tenth grade mathematics, and might well be left for those who specialize further in mathematics or science. It is, however, a genuine problem. The next choice of the newer view would probably be the solution of ". . . and . . . are two curves. Do they intersect? If so, at what points?" This again is not a question which life puts to many persons, but it is one that a sane person need not be ashamed to ask. If genuine applications of the technique of simultaneous equations are beyond the abilities and interests of high school pupils, we may leave it without application until the abilities and interests are available.

### PROBLEMS AS TESTS

Genuine application in the real world should be demanded in problems which are used for training, and is preferable in problems given simply to test the ability to organize a set of statements into equations to answer a question. It is preferable in the later case because, human nature being what it is, teachers will be prone to train for the test. The nature of the examinations used has always influenced the nature of the instruction and probably always will. Except for this, we might permit as algebraic "originals" in tests, the problems about consecutive digits, hands of a watch, numerators and denominators defined by their sums, differences, products and quotients in fantastic ways, and the like, which we exclude from mathematical *training* if better material can be obtained.

### ACTUAL AND DESCRIBED SITUATIONS

We have noted in an earlier article that the ability to manage a problem as encountered in reality, and the ability to manage the same problem as it is described in words in an algebra book need not be identical. Success with the former is consistent with failure with the latter, and *vice versa*.

The worst discrepancies are when, on the one hand, a state of affairs which would be very clear and comprehensible to a person experiencing it, is beclouded and confused by words, and when, on the other hand, the pupil learns to obtain correct solutions by response to verbal cues, the lack of which would cause the real situation, when encountered, to baffle him.

As an example of the first, consider this problem:

"A man is paying for a $300 piano at the rate of $10.00 per month with interest at 6%. Each month he pays *the total interest which has accrued on that month's payment*. How much money, including principal and interest, will he have paid when he has freed himself from the debt?" In reality the man probably would be told that he had to pay $10.05 the first month; $10.10 the second month; $10.15 the third month, etc., and would easily see the progression. The difficulty with the problem in schools lies chiefly in understanding what "each month he pays the total interest which had accrued on that month's payment" means, and in the confusing use of "including principal and interest" and "debt."

It seems wiser to give more attention to providing real situations for the application of algebra. For example, it seems wise for pupils to draw a straight line and another cutting it, and find the size of all four angles by measuring one of them, as well as to solve problems in words about supplementary angles. There is not only a greater surety that the pupils are being prepared to respond effectively to situations which life will actually offer, and an insurance against the danger of unsuitable linguistic demands, but also often an increase of interest in and respect for algebra.

Like almost everything in teaching, we have to add the clause "other things being equal" to this recommendation. Too much time must not be spent in drawing, measuring, weighing and the like. Also, the "real" situation will often be a map already drawn, a table of values already measured, a set of observations already made. Also the genuine problems to which algebra applies are, as compared with those of arithmetic, more often prophetic, foretelling what will happen if certain conditions are fulfilled or what to do in order to bring certain results to pass. The genuine task is, in many of these prophetic problems, precisely to understand a verbal description. Such, for example, are problems about mixtures and alloys, about the amount of $d$ needed to have its proportion to $c$ the same as $b$'s to $a$, and about the drawing of rectangles of specified proportions and total area.

### Isolated and Grouped Problems

Problems grouped by their relation to some aspect of science, industry, business and home life, as *Falling bodies*, or *Alloys*, or *Sliding scales for wages* or *Dietaries* have certain advantages. (1) The situations dealt with are more likely to be understood; (2) things are put together in the pupil's mind that belong together in logic or in reality or in both; (3) the data needed for all the problems can be given once for all, so that in each problem the pupil has to select the facts needed to answer it as well as to arrange them in suitable equations.

The isolated problem is indeed disappearing from arithmetic except in special exercises for particular purposes of tests, reviews, and training in alertness and adaptation. It appears

that by the exercise of enough care and ingenuity, problems in arithmetic can be grouped in this way with no loss to the purely arithmetical training that is given.  The same tendency is operating in algebra, and much good may be expected from it.

## PROBLEMS REQUIRING THE SELECTION OF DATA

The third advantage noted above as characteristic of grouped problems is of special importance, as was noted in an earlier article.

The custom is firmly fixed of giving in a problem only the facts needed to solve it, so that "there are as many distinct statements as there are unknown numbers"; and the pupil is taught to "represent one of the unknown numbers by a letter; then, using all but one of the statements, represent the other unknowns in terms of that same letter.  Using the remaining statement, form an equation."  Yet it seems unjustifiable.  The time and thought now spent by pupils on intricate fabrications whose like they will never see again, might much better be spent in such selective tasks as are genuine and instructive.

## PROBLEMS REQUIRING THE DISCOVERY OF DATA

A further step is worth consideration, namely, that of giving problems some of the data for which are lacking and must be supplied by the pupil's search.  We do this to a slight extent by not including in the statement of a problem such needed facts that 1 foot = 12 inches, or that a square had four equal sides.  Is it desirable to require the pupil to find in his memory or in tables at the end of his textbook on algebra or in other reference books or from observation and measurement such facts as the inter-equivalences of inches and centimeters, the weight of a cubic foot of water, the capacity of a 4 ounce bottle, the length and width of his classroom or the area of Ohio?

There are obvious inconveniences in doing this, but there is the advantage of making problem solving in school one degree more like problem solving in science or industry or business. We might at least go as far as to assign a score of problems each with the question, "What further fact or facts must you have in order to solve this problem?", and distribute the work of discovering these facts among the pupils.  The lesson that one must often supplement the facts given by the situation itself

by further investigations would then be taught to all, at no great cost of time.

Such searching is, of course, not algebra; neither is the understanding of statements about rates, speeds, investments and yields algebra. The algebra begins when statements understood are to be translated into algebraic symbols. Having already far overstepped that line in the customary work with verbal problems, we may go farther with no inconsistency.

### PROBLEMS REQUIRING GENERAL SOLUTIONS

. The most objectionable feature of problem solving in algebra today to a psychologist is the predominance of problems seeking a particular fact about some particular state of affairs—the relative neglect of problems which seek the general relation between variations in one thing and variations in something related to it.

The main service of algebra, as the psychologist sees it, is to teach pupils that we can frame general rules for operating so as to secure the answer to *any* problem of a certain sort, and express these rules with admirable brevity and clearness by literal symbolism. We take great pains to teach the pupil that $pq$ means the product of whatever number we let $p$ equal and what ever number we let $q$ equal; and that if $p$ and $q$ equal any two numbers, the first number times the product of the two equals $p^2q$, and other similar facts. Then, in problems, the $p$'s and $q$'s or $x$'s and $y$'s in nine cases out of ten, mean something as unlike "any number" as could possibly be. We build up habits of computing with literal numbers and then, in problems, make almost no use thereof, reverting to an arithmetic plus negative numerals with a written $x$ in place of the mental "What I am trying to find." Small wonder that the pupil often thinks of his algebraic computations as a mere game that one plays with $a$, $b$, $c$, $d$, $+$, $-$, $\times$, $\div$, and (). If, after a few exercises in the use of letters to mean "any number of so and so," and a few exercises in reading and framing formulae, we have him do nothing with literal symbols but play computing games, why should he think otherwise?

Why should we blow hot and cold in this way, asserting that algebra teaches us what is true of any number and then, in

problems, making its linear equations true of only a single number, and its quadratics of only two? Should we not alter many of our IIBb problems into the IIBa form, requiring the pupil to frame the general equations or formulae to solve any problem of that sort, and to obtain any particular answer by evaluating? For example, compare the two tasks I and II below:

I. A man has a lawn 40 ft. long and 30 ft. wide. How wide a strip must he mow beginning at the outside edge in order to mow half of it?

II. 1. A man has a rectangular lawn. Make a formula to state how wide a strip he must mow beginning at the outside edge in order to mow half of it. Let $l$ and $w$ equal the length and width of the lawn in feet.

Let $s$ equal the width of the strip in feet.

2. Find $s$ if $l = 40$ and $w = 30$.
3. Find $s$ if $l = 100$ and $w = 20$.
4. Find $s$ if $l = 80$ and $w = 40$.
5. Find $s$ if $l = 80$ and $w = 60$.

It seems reasonable to progress from problems of the I type to problems of the II type just as we progress from numbers to letters, and from such facts as $2 \times 2 = 2^2$, or $3 \times 3 = 3^2$ to such facts as $a \times a = a^2$, or $a(b + c) = ab + ac$.

Amongst problems requiring a general solution in terms of a literal formula, special importance attaches to problems of direct and inverse proportion, problems where one number varies as the square or square root of another, and other problems involving linear, hyperbolic and parabolic relations.

### Problems of Puzzle and Mystery

The earliest problems of algebra were problems of puzzle and mystery, such as Ahmes' "A hau, its seventh, it equals 18," or the finding of the age of Diophantus from his epitaph. "Diophantus passed one-sixth of his life in childhood, one-twelfth in youth and one-seventh more as a bachelor; five years after his marriage a son was born, who died four years before his father did at half the age at which his father died."

Such problems make an appeal to certain human interests. Some pupils doubtless prefer them to straight-forward uses of algebra in answering questions of ordinary life. The human tendency to enjoy doing what we can do well, and especially

what we can do better than others can, is often stronger than the tendency to enjoy doing what we know will profit us. Some of these problems are also arranged as strong stimuli to thought for thought's sake. By introducing an element of humor they may relieve the general tension of algebraic work, as is at times desirable. They are much more appropriate in algebra for the selected group of superior pupils who continue to high school than they are in arithmetic for all children. On the whole, however, the ordinary applications of algebra to science, industry, business and the home will give better training to the general run of high school freshmen and will inspire greater liking and respect for mathematics than will these appeals to the interest in puzzles and mystery.

One of the best forms of appeals to the puzzle interest is by abstract problems such as: "When will $a^2$ be less than $a$?" "When will $l$ divided by $a$ be greater than $a$?" "State a condition such that $\frac{a}{b}$ will equal $\frac{b}{a}$?" "State a condition such that $abc$ will equal $a$?"

One of the best forms of appeal to the interest in mystery is to have pupils frame formulas* for such mysteries as: "Think of any number and I will tell you what it is. Think of the number. Add 3 to it. Multiply the result by 7. Subtract 20. Tell me the result. The number you thought was . . . (This result diminished by 1 and then divided by 7)." They may also make up such mysteries for the class, score being kept of the time required for pupils to find the formula for the mystery and penalties being attached to devising a "mystery" that doesn't work. $(a + b)(a - b) = a^2 - b^2$ may be taught as a mystery for quickly computing products like $2998 \times 3002$, or $4980 \times 5020$. The formula for the sum of an arithmetic progression may be taught as a mystery for computing the sums of such series, either complete, or in the form "All the numbers from . . . to . . . except . . . and . . ." There are, however, better motives to use for mastering

$$S = \frac{n}{2} [2a + (n - l)d]$$

As has been so often insisted, the cardinal sin in connection

with problems of puzzle and mystery is their decoration with a description of conditions and events in nature which makes the pretence that the problem is genuine when it is not; and so confuses and debauches the pupil's ideas of the uses of algebra. If they are presented in their true light and if the pupils have the option of solving them or solving problems of genuine application,—they can at the worst, do very little harm.

### The Election of Problems by Students

Many of the difficulties of teaching in the case of problems are greatly lessened by arranging to have each choose a certain number of problems to solve from a list which contains, say, five times as many as any one pupil is to solve. We have just noted the value of election between "useful" and "puzzle" problems, if the latter are presented at all. We have noted previously that problems drawn from physics may be of very different value to pupils who are studying general science and to pupils who are not. Within the latter group, we might also differentiate between those who happen to be ignorant of science, and those who are so by their own volition. Boys and girls may well differ in their choices, though probably not so much as some theorists would expect. It may be desirable to permit and even encourage some pupils to choose the easier problems.* The provision of five times the number of verbal problems now given in standard textbooks would add perhaps two cents to the cost of production.

### Summary

It is a worthy aim to teach pupils to organize the facts of important situations requiring numerical responses into equation form and to solve their equations. It is also worth while for pupils to learn that any quantitative question, no matter how elaborate and intricate, can be so expressed, provided adequate data are given.

Even if the educative value of this work is improved by such modifications as have been suggested in this chapter, it will still be, on the whole, less important than the framing of general equations or formulas for solving *any* problem of a certain kind. Learning to let $x$ or $q$ equal the unknown and to express data

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*The instructions may be "Do the ten hardest ones that you think you can do."

in terms of their relations to it is a useful lesson, but learning
to express a set of relations in generalized form is a more use-
ful one, and, so far as psychology can prophesy, one more likely
to transfer its improvement to other abilities.  It is when the
verbal problems of algebra advance beyond arithmetical prob-
lems in the same way that algebraic computation advances be-
yond arithmetical computation that they perform their chief
educational service.

# THE TEACHING OF BEGINNING GEOMETRY

By A. J. SCHWARTZ
Cleveland High School, St. Louis, Mo.

It seems necessary before presenting the main points of this paper to discuss, briefly, some of the underlying principles on which they are based. I ask your indulgence, therefore, if I appear to go somewhat far afield in my search for an adequate setting.

Geometry is essentially the study of spatial relations. "Of all forms of relational consciousness its concepts are the most directly perceived and familiar facts in human experience," says the philosopher, Ernest Mach. "The only other relational consciousness which is anything like as familiar is time, and these are the two universal forms in which all sensory facts appear in experience. . . . The first geometrical knowledge was acquired accidentally and without design by way of practical experience, and in connection with the most varied employment." Hence, in its fundamental aspects, the science of geometry differs from the science of physics only in the respect that mental experiment can be performed therein with far simpler experiences and with such as have been more easily and almost unconsciously acquired. In the illuminating language of Poincaré, "Geometry is not true, it is advantageous. . . . By natural selection our mind has adapted itself to the conditions of the external world." Our notions of space are, therefore, "rooted in our physiological organism"; so that, "without the least artificial or scientific assistance we acquire abundant experience of space. We can judge approximately whether rigid bodies which we perceive along side one another in different positions at diffrent distances will, when brought successively into the same positions, produce approximately the same or dissimilar space sensations. We know fairly well whether one body will coincide with another, whether a pole lying on the ground will reach to a certain height, etc." "It is extremely probable," says Mach, "that the experiences of the visual sense were the cause of the rapidity with which geometry developed"; for, "visual space forms the clearest, precisest and broadest system of space sensations."

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"From the historical point of view," says Judd, in his *Psychology of High School Subjects,* "one would expect geometry to come early in the course, in view of the fact that it matured earlier than algebra. The fact is," continuing the quotation, "its very perfection served to take geometry into the highest schools. In the University of Alexandria the results of Greek studies of space were put into logical form by Euclid. This logical form was also borrowed from Greece, where Aristotle had evolved that perfect system of syllogistic logic which dominated the whole period of mediæval thought. A complete geometry expressed in perfectly logical form became one of the highest courses of study. Studies of space were no longer of the primitive type that had grown up among the early Greeks." So well did the Greeks, in the critical period of Greek mathematics, attain their ideal—"to systematize the knowledge of their province for professional and scholarly contemplation and for purposes of individual survey, to separate what was directly cognizable from what was deducible, and to throw into distinct relief the thread of deduction," that today geometry is no longer recognized as a separate science, distinct from mathematics, as is physics or chemistry.

It appears manifest, therefore, that the traditional difficulties of beginning geometry are not traceable to the spatial conceptional material, but must be sought elsewhere. Poincaire, in his essay on "The Philosophy of Science," offers the following corrective to the difficulties: "Zoologists maintain that the embrionic development of an animal recapitulates in brief the the whole history of its ancestors throughout geologic time. The teacher should make the child go over the path his father trod; more rapidly but without skipping stations. For this reason," he adds, "the history of science should be our first guide." The same idea is tersely stated by Mach as follows: "An idea is best made the possession of the learner by the method by which it has been found." From Felix Klein we have the following advice: "In teaching it is not only admissable, but absolutely necessary to be less abstract at the start, to have constant regard to the applications, and to refer to the refinements only gradually as the student becomes able to understand them. . . . : If, too, much emphasis is placed on rigor at the beginning a large part of the subject will remain unintelligible."

In the light of this wholesome advice, and in order to give our setting an historical background, let us trace, briefly, the development of geometry from antiquity. The Greeks inherited the nucleus of classified geometric knowledge from the Egyptians. The total amount of this information consisted of a few practical rules, some of which were little better than mere guesses, for example, their rule for finding the area of a quadrilateral, namely, the product of the arthmetical means of the opposite sides: Perhaps the most celebrated rule which the Egyptians handed down to the Greeks was the one for measuring off a right angle by means of constructing a right triangle whose sides are in the ratio 3:4:5. Regarding geometry as a science, the Egyptians had not even made a beginning, and, says the historian, "there are reasons for thinking that Egyptian knowledge remained virtually stationary for at least two thousand years prior to their contact with the Greeks."

"The founder of the earliest Greek school of mathematics and philosophy was Thales, one of the seven sages of Greece, who was born about 640 B. C. at Miletus, and died in the same town about 550 B. C. We cannot form an exact idea as to how Thales presented his geometrical teaching," says the historian. "We infer, however, from Proclus that it consisted of a number of isolated propositions which were not arranged in a logical sequence, but that the proofs were deductive, so that the theorems were not a mere statement of an induction from a large number of special instances, as probably was the case with the Egyptian geometricians. The deductive character which he thus gave to the science is his chief claim to distinction. The following comprise the chief propositions that can now with reasonable probability be attributed to him; they are concerned with the geometry of angles and straight lines.

"(1) The angles at the base of an isosceles triangle are equal. . . . Proclus seems to imply that this was proved by taking another exactly equal isosceles triangle, turning it over and then superposing it on the first—a sort of experimental demonstration.

"(2) If two lines cut one another, the vertically opposite angles are equal. Thales may have regarded this as obvious, for Proclus adds that Euclid was the first to give a strict proof of it.

''(3)  A triangle is determined if its base and base angles are given.

''(4)  The sides of an equiangular triangle are proportional.

''(5)  A circle is bisected by any diameter.  This may have been enunciated by Thales, but it must have been recognized as an obvious fact from the earliest times.

''(6)  The angle subtended by a diameter of a circle at any point in the circumference is a right angle.  . . . .  This appears to have been regarded as the most remarkable of the geometrical achievements of Thales, and  . . .  it has been conjectured that he may have come to this conclusion by noting that the diagonals of a rectangle bisect each other, and that therefore a rectangle can be inscribed in a circle.  If so,'' continues the historian, ''and if he went on to apply proposition (1), he would have discovered that the sum of the angles of a right-angled triangle is equal to two right angles, a fact with which it is believed he was acquainted.  It has been remarked that the shape of the tiles used in paving floors may have suggested these results. . . .''  Commenting further, the historian suggests ''that it is possible he was also aware, as suggested by some modern commentators, that the sum of the angles of any triangle is equal to two right angles.  As far as equilateral and right-angled triangles are concerned, we know from Eudemus that the first geometers proved the general property separately for three species of triangles, and it is not unlikely that they proved it thus.  The area about a point can be filled by the angles of six equilateral triangles or tiles, hence the proposition is true for an equilateral triangle.  Again, any two equal right-angled triangles can be placed in juxtaposition so as to form a rectangle, the sum of whose angles is four right angles; hence, the proposition is true for a right-angled triangle.  Lastly, any triangle can be split up into the sum of two right-angled triangles by drawing a perpendicular from the biggest angle on the opposite side, and therefore again the proposition is true.  The first of these proofs is included in the last, but there is nothing improbable in the suggestion that the early Greeks continued to teach the first proposition in the form above given.''

I have quoted the historical comments here at some length because of their relevancy to primitive deductive methods.  It is evident that the results were arrived at by a kind of ''sensuous

mental experimentation with figures." Hence, if the history of geometry is to be a trustworthy guide, we should be led to indorse the ideas of Henrici, that in the beginning the teacher should put the subject in such a manner that (the student) may realize the geometrical contents of the propositions as properties of space through actually seeing their truth by the mental or physical inspection of figures, instead of being convinced of their truth by a long process of logical reasoning.

Following Thales, the next great Greek geometer to advance the science was Pythagoras who flourished from about 569 B. C. to about 500 B. C., and " 'who,' says Proclus, quoting from Eudemus, 'changed the study of geometry into the form of a liberal education, for he examined its principles to the bottom and investigated the theorems in an  . . .  intellectual manner.' " Says Ball, "Pythagoras probably knew and taught the substance of what is contained in the first two books of Euclid about parallels, triangles, and parallelograms, and was acquainted with a few other isolated theorems including some elementary propositions on irrational magnitudes; but it is suspected that many of his proofs were not rigorous, and in particular that the converse of a theorem was sometimes assumed without proof."

A point worth noting in this connection is the fact that Pythagoras and his contemporaries, mentally keen though they were, were guilty of ignoring many logical refinements which we are sometimes reluctant to permit even during the first few weeks on the part of rather young pupils.

Following Pythagoras was a period of about 200 years when Greek philosophy was rife with sophists and paradox jugglers. These latter, says Ball, "made the Greeks look with suspicion on the use of infinitesimals," and may we not also add, probably influenced Euclid in his evasive use of motion, when a free use of the latter would have added much to the clearness of his exposition.

Undoubtedly the two most influential factors in bringing about the extreme formalism of Greek geometry were the two great philosophers, Plato and Aristotle, who lived in the century preceding Euclid. Primarily not mathematicians in the sense of being original investigators, "the latter," says Ball, "was chiefly concerned with mathematics and mathematical

physics as supplying illustrations of correct reasoning; and to the latter is due that subsequent geometricians began the subject with a carefully compiled series of definitions, postulates, and axioms.''

Commenting on these historical notes, we see that unconscious intuition played an extremely important part in early Greek geometry. ''Indeed,'' says Klein, ''history reveals in every case that unconscious intuition has been especially active during the genesis of every important advance in the development of mathematics.'' We see that in its early stages, the process of deduction was little more than a sort of visual inferential process based on placing figures in certain favorable positions, either by accident or design. Pertinent to this point is the following excerpt from his essay on ''The Philosophy of Science,'' by Poincaré:

''The principal aim of mathematical teaching is to develop certain faculties of the mind, and among them intuition is not the least precious. It is through it that the mathematical world remains in contact with the real world, and if pure mathematics could do without it, it would always be necessary to have recourse to it to fill up the chasm which separates the symbol from reality. For the pure geometer himself, this faculty is necessary; it is by logic one demonstrates, by intuition one invents. To know how to criticize is good, to know how to create is better. . . . Logic tells us that on such and such a way we are sure not to meet an obstacle; it does not say which way leads to the end. For that it is necessary to see the end from afar, and the faculty which teaches us to see is intuition. Without it a geometer would be like a writer who should be versed in grammar but had no ideas. Now how could this faculty develop, if as soon as it showed itself we chase it away before knowing the good of it? . . . In the exposition of first principles . . . it is necessary to avoid too much subtility; there it would be most discouraging and moreover useless. We cannot prove everything and we cannot define everything; and it will always be necessary to borrow from intuition; and what does it matter whether it be done a little sooner or a little later, provided that in using correctly premises it has furnished us we learn to reason soundly. . . . The essential thing is to reason soundly on the assumptions admitted.''

But intuition is not exact, and this is the point where the hypercritical logician inserts his objections. He abhors such statements as "geometry is concrete," unmindful of the fact that we cannot think except in terms of imagery which has a degree of objectivity, for "only the actually visualized or imaged figures can tell us what particular concepts are employed in a given case." Klein illustrates this graphically as folows: "Newton," he says, "assumes without hesitation the existence in every case of a velocity of a moving point, without troubling himself with the inquiry whether there might not be continuous functions having no derivatives. Thus," he continues, "when thinking of a point, we do not think of an abstract point, but for something concrete for it. . . . In imaging a line, we do not picture to ourselves 'length without breadth,' but a strip of certain width. Now such a strip has, of course, always a tangent; that is, we can always imagine a strip having a small portion (element) in common with the curved strip. . . . The definition in this case is regarded as holding only approximately, or as far as may be necessary."

Reverting again to the historical references in order to complete the review, we note that the extreme formalism of Greek geometry was a comparatively late development, reaching its present state of perfection fully 300 years after the first school of Greek mathematics; and largely induced by the influence of professional logicians, not mathematicians, whose principal aim was to produce philosophers, and not geometers.

Therefore, to be in harmony with the lessons of history, let us begin geometry in a simple, natural manner, giving free play to the intuition. Let us postpone formal methods until the mind of the pupil is familiarized and matured by a sufficient frequentation with informal geometric reasoning. Let us not begin our subject by extracting all objectivity from the notion of planes, lines and points, but let us tacitly regard the planes as "corporeal sheets," the line as "a thread vanishingly small in thickness," and a point as a small "corporeal space." "Nothing prevents our idealizing in the usual manner these images at the proper time by simply leaving out of account the thickness of the sheets and threads." For our introduction let us present a schematized progressive development which shall lead the pupil gradually, by easy stages, to an appreciation of formal

deductive methods. In this preparatory course, let us avoid the technical language of formal logic, such as hypothesis and conclusion; and above all, let us not insult the intelligence of our pupils, or mystify them ''by telling them that they do not understand what they think they understand, and that they must prove what seems to them evident.'' In this progressive development, let us develop the properties of the straight line and the circle together, because each throws light on the other, and because these properties can very frequently be inferred immediately by a mental inspection of the figure; and, thirdly, because many of the propositions can be immediately applied in geometric drawing where they will seem useful to the student.

Let us, therefore, begin geometry with ruler and compasses, forth with. From the tests of the straight edge will arise the fundamental properties of the straight line; the elementary properties of the circle will follow quite as readily. These being obvious intuitions, may be passed over with simple acquiescence on the part of the pupil. Simple exercises in drawing may then be given, always having in mind the encouraging of thought processes, chiefly intuitional at this stage, and not simply mechanical manipulation. I prefer to stress and encourage independent thinking from the first day, for ''the initial method will largely determine the subsequent mental attitude of the pupil toward a subject, no matter what may afterward be done.''

A few simple examples with concrete settings, and calling for the exercise of clear visualization will be appropriate. As an example, I mention the following which interested me greatly when I was a boy, and which was contained in an old arithmetic, I think it was Ray's: ''A spider in an upper corner of a square room of given dimentions observes a fly in the lower diagonally opposite corner. If the spider proceeds to attack his prey by the most direct route, traveling on the walls and ceiling, or floor, what will be his path?'' With some discussion followed by a few suggestions, including the hint that the room may be represented by a cardboard box, a solution will be forth coming. The solution which consists in flattening out the walls of the room into the plane of the floor, and joining the original position of the spider with that of the fly will not only be convincing, but it will provide the opportunity to drive home the

point that a very obvious property of the straight line is the key to the solution which is not at all obvious. This will promote respect for summaries of obvious properties of figures.

Our next important consideration is that of motion. Without doubt motion should play a larger part, and should receive much more emphasis in elementary geometry than it has in the past. In his essay on "The Philosophy of Science," Poincaré points out that "geometry (viewed from the standpoint of higher analysis) is the study of a group; that of the motion of solid bodies"—as a side light bearing on another point, he says, "How define this group then without moving some solids." In another vein, Mach remarks, "The study of the movements of rigid bodies, which Euclid studiously avoids and only covertly introduces in his principle of congruence, is to this day the device best adapted to instruction in elementary geometry." The notion of congruence is of course inseparable from the idea of motion. Instead of slurring over the motions involved in superposition, they should be stressed. All the more so because intuition plays a big part here, and, if we do not chase it away, referring to the admonition of Poincaré, it will help us over most of the difficulties. The real difficulties underlying these proofs as usually given, are due to the logical technique employed, and which ought to be postponed, and to certain subtilities implied in the obvious properties of the straight line. If, however, the motions are stressed, the logical technique is postponed, and subtle refinements are kept in the back ground, a clear comprehension of the basic principle of congruence will result. If the teacher's preliminary questions are skilfully formed the logical difficulties may be avoided. Thus, after having discussed the three kinds of motion; sliding motion or translation, rotation about a point, and revolving about a straight line as an axis, the teacher may safely ask the following questions: How may the line segment $AB$ be brought into coincidence with the equal segment $CD$, employing several examples in which $AB$ and $CD$ have various relative positions? How may a circle be brought into coincidence with a congruent circle? How may an arc of a circle be brought into coincidence with an equal arc? How may the letter $L$ be brought into coincidence with the inverted $L$? And so on. All of these questions will be answered intuitively with little hesitation. If, however, one asks why the seg-

ment $AB$ coincides with the segment $CD$ after $A$ falls on $C$ and $B$ falls on $D$, he will get no answer, or if he does get an answer, it will probably be because the segments are equal. This being an intuitive fact, the student can at this time have no reason to doubt it; to doubt intelligently, "one must know why one doubts." In other words, the fundamental property of the straight line on which the answer depends is a useless refinement at this stage of progress.
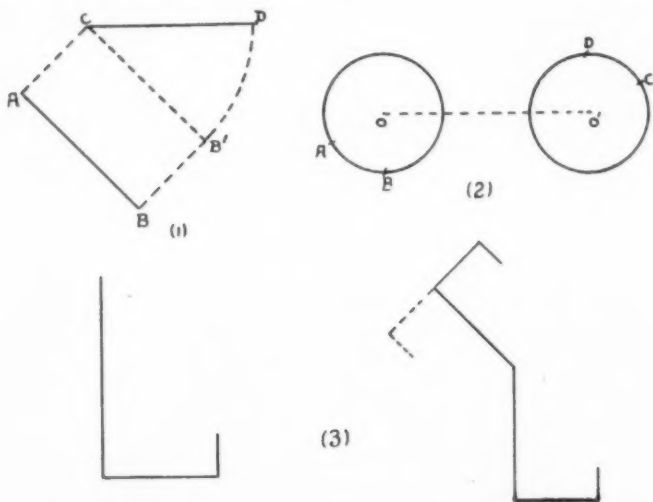


Thus, referring to the illustrations (see Fig. 1) $AB$ is brought into coincidence with $CA$, by first sliding $AB$ along the line $AC$ until $A$ coincides with $C$; then by rotating $AB$ about $C$ as a center. That is, a line segment can always be brought into coincidence with an equal segment, in the same plane, by a slide followed by a rotation. Similarly, the circle $O$ may be brought into coincidence with the circle $O'$ by sliding the former along the line $OO'$ until the centers coincide; also the arc $AB$ may be brought into coincidence with the equal arc $CD$ by a slide (that is, by bringing the circles into coincidence) followed by a rotation.

Likewise, it is easily visualized that the upright $L$ may be brought into coincidence with the inverted $L$ by applying consecutively the three types of motion—sliding, rotating and turning (or revolving) motion.

We are now in a position to introduce a principle which will further our schematization and at the same time parallel, in a general way, the methods employed by the contemporaries of Thales—the principle of symmetry. "The circle being the embodiment of symmetry," adopting the phraseology of Henrici, by studying its relations to the straight line in certain favorable positions, we shall be enabled to derive a variety of useful properties by a simple mental inspection of figures.

An ink blot thrown on a scrap of paper which latter is folded through the blot while the ink is still wet will instantly command the attention of the class. There results a grotesque figure to be sure, the form of a beetle's shadow; nevertheless, it exhibits the desired properties, and it is a form which the pupil recognizes as widely distributed in nature. A little discussion will bring out the obvious properties of a symmetric figure.
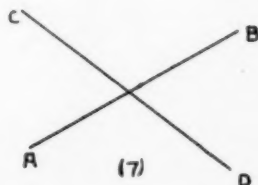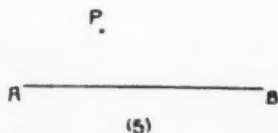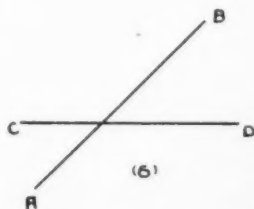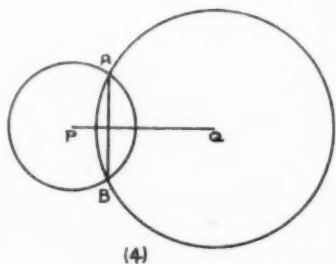


Having defined symmetry with respect to an axis, the fact that a circle is symmetric with respect to any diameter follows immediately as an intuitive fact. It follows not less clearly that two intersecting circles are symmetric with respect to their line of centers. Whence (see Fig. 4), by a mental inspection of the figure we infer immediately that the common chord is bisected by the line of centers. A judicious line of questions will now

readily lead to the discovery of the usual method of bisecting a line segment with compasses and ruler. Following the above, a variety of simple exercises may be given, such as: (Fig 5) Draw a straight line $AB$ and an external point $P$; find a point $Q$ which corresponds to $P$ in the symmetric figure of which $AB$ is an axis. (Fig. 6) Draw two straight lines $AB$ and $CD$; find a straight line which corresponds to $AB$ in the symmetric figure of which $CD$ is the axis. (Fig. 7) Draw two intersecting straight lines $AB$ and $CD$; find the axis of symmetry of the figure in which $AB$ and $CD$ are corresponding lines. This, of course, is the familiar problem, "To bisect an angle," expressed in different language; when it appears later in the usual form its mastery by the pupils will require little effort. Simple problems based on architectural drawing may also be assigned to provide concrete applications.

The solution of the first problem, referring to Fig. 5, results from drawing two circles passing through $P$ and having their centers on $AB$.

In the second problem, Fig. 6, it is obvious that the line corresponding to $AB$ must pass through $O$, the point where $AB$ cuts $CD$. The solution, therefore, depends on obtaining a second point lying on the required line. Hence, taking any point on $AB$, and repeating the construction of problem 1, the solution is obtained.

If now we define perpendicular to a line as a line symmetric to the former as an axis, we may easily solve the usual problems for constructing perpendiculars from a given point to a given straight line. After a few well chosen questions referring to the symmetry of two intersecting circles, the pupils will readily discover or invent the constructions (see Figs. 6 and 9). Moreover, by a simple inspection of the figure resulting from the symmetry of two intersecting circles, we easily infer the following important proposition: "Every point on the perpendicular bisector of a line segment is equally distant from the end points of the segment" (Fig. 10). This proposition enables us to find the center of a circle when three points on the circle are given, and other related problems. The student who takes part in solving problems of this kind begins to have awaken within him an appreciation of, and for, geometric reasoning. To foster this growing spirit, I habitually propose a problem in this connec-

tion (there are doubtless better ones) whose solution is not at all obvious, but the proof of which will be welcome, and which will carry real conviction: In Fig. 11, $A$ represents the position of a house, $B$ that of a barn, and $MN$ the edge of a river. Required the position of a pumping station, located on the river bank, so that the water can be piped to both house and barn with the least length of pipe. The essentials of the problem will be recognized as being the same as those of the familiar

P.

A ———————————— B
(8)

A ————— P ————— B
(9)

(10)

(11)

(12)

(13)
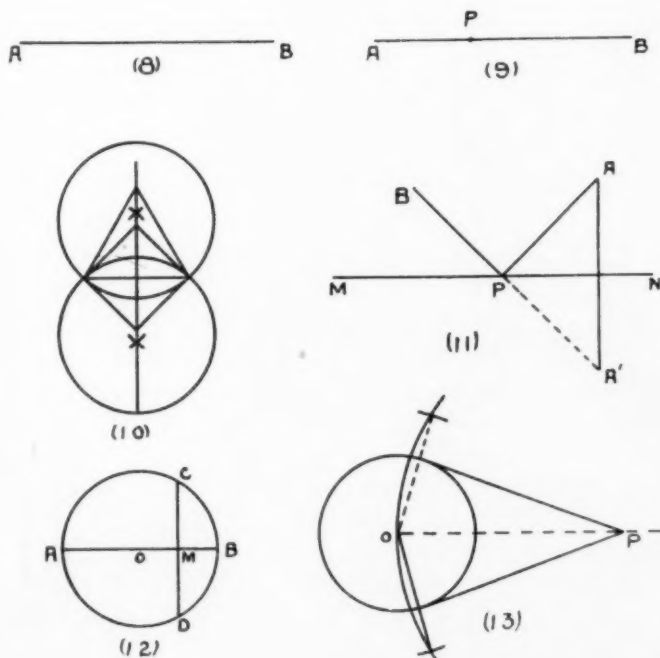
problem concerning the reflection of light—a more objective setting being substituted. The solution of the problem involves nothing more than the proposition concerning the perpendicular bisector of a line segment, just derived, and the proposition that the shortest path between two points is the line segments joining them. The problem is, of course, too difficult to assign for independent work; however, it yields readily to solution under the

eyes of the teacher. The only object in solving it with the pupils is to cultivate an appreciation for a geometric proof differing from a direct inference. This I think it does.

The limits of this paper forbid my developing in too great detail a complete course leading up to formal methods. Suffice it to say, that other constructions may be developed in an analogous manner. Thus, by carrying out in detail the analysis of the motions by which an arc is brought into coincidence with an equal arc, the discussion invoked leads naturally and readily to the relations implicating the corresponding chords and central angles. It is then but a step, and intuition will lead the way to the solution of the problem, to construct an angle equal to a given angle.

The developments leading up to the construction of tangents to a circle will serve to give variety to the illustrations, and may perhaps merit a more detailed description. Consider any circle with center $O$ (Fig 12). Draw any diameter $AB$, and a chord $CD$ perpendicular to $AB$. From the symmetry of the figure, remembering that a perpendicular to a straight line is symmetric with respect to the latter as an axis, we infer immediately by an inspection of the figure that the chord is bisected by the diameter. The obvious inference that a diameter which bisects a chord is perpendicular to it, and that the perpendicular bisector of a chord passes through the center of the circle will also be noted and summarized. We are now prepared to consider the notion of a tangent. Observing that a straight line whose distance from the center is greater than a radius has no point in common with the circle, and that a straight line whose distance from the center is less than a radius cuts the circle in two points, we are led to inquire as to what must be the relative position of a straight line with respect to the circle, when its distance from the center is exactly equal to the radius. The reasoning may then be given that, theoretically, the line can have but one point in common with the circle. The reasoning will be convincing, and it will be welcome, because it proves something that baffles the visual sense. The method for constructing a tangent to a circle at a given point on the circle will be obvious.

The method for constructing a tangent to a circle from a point outside the circle can then be readily developed. Analyzing the

problem, we note that the required line must be perpendicular
to the radius drawn to the point of contact.  Leading questions
referring to the obvious properties of the figure studied at the
beginning of the topic bring out the details of the construction
(Fig. 13), which consists in describing a circle passing through
the center of the given circle and having its center at the given
external point $P$.  The point of contact will then be where the
chord from $O$, and equal to the diameter of the given circle,
intersects the latter.  It is scarcely necessary to remark that the
usual properties implicated in the figure resulting from the
construction of the tangents to a circle from an external point
may be inferred immediately from symmetry by a simple inspec-
tion of the figure.

The topic on tangents provides a variety of exercises, both
concrete and abstract, for independent construction work by the
pupils.  Exercises are also available for making simple infer-
ences, other than from visual inspection of figures.  As examples
of the latter may be noted the following: "If two circles are
tangent to each other, the tangents drawn to them from any
point of their common tangent are equal."  "If two circles are
tangent to each other, externally, their common interior tangent
bisects their common exterior tangent."

Informal work of this kind could be prolonged indefinitely.
It is a matter of experience and judgment on the part of the
teacher as to how much time may be profitably spent in this
manner.  But let there be no mistake about it, time is not wasted
by alloting a reasonable period for this sort of a preparation,
and there should be no more failures than in the average of
other academic subjects.

Let us suppose that three weeks have now elapsed, and that
we are prepared to adopt formal methods of procedure.  Let
us not, however, fall into the error of thinking that we can begin
formal geometry in the technical sense of the term; that is not
possible in the high school, least of all at the end of only three
weeks of an informal introduction.  We can, of course, adopt
formal language, but we can only approximate formal methods of
deduction.  Some discussion of logical terms and the nature of
formal statement of proof will be necessary.  Having covered
this ground, amplifying the explanations with illustrations
chosen from the material developed informally, we may proceed

with the topic of congruent triangles. I prefer to approach
the topic thus:

Given certain dimensions, construct a triangle.

(1) When two sides and the included angle are given.

(2) When two angles and the included side are given.

(3) When the three sides are given.

(The pupils are, of course, familiar with the methods for
constructing equal angles, and with the use of the protractor,
for that matter.)

Then, having constructed two triangles with two angles and
the included side of the one equal, respectively, to two angles
and the included side of the other, I provoke a discussion as to
how either triangle may be brought into coincidence with the
other, arranged in various relative positions. The motions are
analyzed thus: Slide triangle $ABC$ along the line $AA'$
until $A$ coincides with $A'$; then rotate triangle $ABC$ about $A'$ as
a center until $AB$ coincides with its equal $A'B'$; then (assuming
that we are dealing with the most general relative positions)
turn triangle $ABC$ over on $A'B'$ as an axis. When $AC$ will fall
on $A'C'$, and $BC$ will fall on $B'C'$. The reason for the latter
being correctly given, the conclusion that $C$ falls on $C'$ is reached
and justified, visually by the pupils, and logically by the teacher.
Theorem (1) having been similarly treated, the formal state-
ments of these proofs as presented in the text are discussed.
The pupils are made to see that, save for the language, the
proofs are essentially the same.

Henceforth, for some little time, facility in formal statement
of proof is the immediate aim. The topic of congruent triangles,
as every experienced teacher knows, furnishes the very best drill
for this purpose; not by reading or memorizing the standard
propositions exhibited in the text, but by proving simple exer-
cises depending on the basal theorems proved above. For, as has
been well said by Schulze, ''the solution of a large number of
simple exercises will enable the student to understand clearly
and to appreciate model demonstrations.''

Time forbids further detailed discussion of methods of pres-
entation. In passing, however, I may remark, that little text
book advance should be made until the pupils are fairly effi-
cient in proving simple exercises of this kind. In this connec-

tion, I wish to remark in a general way, that much harm may
be done in the beginning by over-emphasizing the recitation
method of teaching, especially if the teacher exacts fluent and
complete demonstrations.  Thus to expect a pupil early in the
course to study an average text book demonstration out of class;
then on the morrow put the figure on the board, complete in
every detail, and, with pointer in hand, rise and give the entire
proof, is to approach geometry by the wrong method.  The re-
sults cannot but be fatal to a large percentage of the class.  If
one assigns a proposition for home study, early in the course,
a much better practice in recitation is to put the figure on the
board, without auxiliary construction lines; then, writing the
essential steps on the board systematically, develop the proof,
step by step, calling on this pupil, then that pupil, adding con-
struction lines when necessary at the time they are needed.
After the proof has been fully developed, a single pupil may be
asked to give it in full; or, the same may be asked the next day,
but certainly not at the first reading.

It cannot be too strongly emphasized that real power in
geometry comes from practice in mastering original exercises.
Hence, for the first ten or twelve weeks the teacher should be
satisfied with a very imperfect display of text book knowledge.
He should bend his energies toward developing power in master-
ing originals, even very simple ones; in the end time will be
saved, for ability in mastering originals gives a power of com-
prehension that will make text book theorems easy reading.
After a student has discovered himself, so to speak, the most
exacting recitation methods may be employed with perfect
safety.

Since I have emphasized the importance of exercises, perhaps
in order not to be misunderstood, I should say something as a
counter-balance.  It is a curious fact that in Legendre's geom-
etry, a little less than a hundred years ago, there was not a
single exercise; and as late as 1877 Chauvenet's geometry had
the following paragraph in the preface: "As the student can
make no solid acquisition in geometry without frequent practice
in the application of the principles he has acquired, a copious
collection of exercises is given in the appendix," etc.  Note the
word *appendix!*  In other words, the only exercises found in the

book were at the end.  A little later, as you know, the exercises
appeared at the end of the various books, or divisions of the
texts; and now they are scattered throughout the text.  It ap-
pears therefore, that exercises came into geometry by the back
door and have only recently reached the front door.  Will they
be content with a supplementary role, or will they, like true
revolutionaries, attempt to usurp the whole stage?  I hardly
think there is danger of the latter contingency, though I do
think there is danger of overloading a text with twaddle, so that
marily with the rates of such changes and with the determination
the fundamentals will not appear in proper relief.  It seems to
me important that fundamentals and exercises should be shown
in their true relations, and that the latter should always be
employed to clarify and to aid in gaining a clear understanding
of the former.  Exercises should, therefore, be significant, they
should have a point, and not result in mere busy work.  Thus,
for example, the insertion of page after page of preliminary
work concerning the computation of complimentary and supple-
mentary angles, and of purely mechanical constructions, involv-
ing patterns of various designs, etc., is largely a waste of time.
The so-called geometers or ''rope stretchers of Egypt'' con-
structed right angles, performed computations and probably
drew designs for at least twenty centuries without making any
appreciable advance in the science of geometry, and it isn't
probable that our pupils will profit in any greater degree in two
or three weeks' experience with the same kind of material.

# THE ORIGIN OF MATHEMATICS—A FIRST LESSON IN SECONDARY MATHEMATICS

By WILLIAM BETZ

East High School, Rochester, N. Y.

*Introduction:* The nineteenth century was characterized by an unparalleled activity in the realm of science. When Darwin and his followers succeeded in popularizing the doctrine of evolution, an amazing impetus was given to research in almost every field of human thought and endeavor. It soon became natural to go back to first principles, to study the growth and development of all existing things, to look for causes, motives, connections, controlling forces. New sciences sprang up over night. Innumerable questions were addressed to nature by carefully arranged experiments. The bulk of scientific knowledge increased immensely. And, as if by magic, even the past gave up many of its long concealed secrets. The geologic record of the rocks became legible. It revealed upheavals extending through aeons of time. The incredibly slow unfolding of life on this globe became established by countless fossilized remains. It was found that human civilization, too, had a long, prehistoric career. The story of man's painful upward struggle is still very fragmentary, but new data are coming to light every year. Many of our most precious human achievements are now known to have their roots in this early prehistoric period. It is becoming clear that many arts and sciences owe their origin to universal stimuli and to fundamental human needs. And when man finally succeeded in leaving written records on stone or papyrus, an almost modern state of mentality had been reached. The human mind, says a recognized scholar, has not changed essentially since palaeolithic time.

And now, Babylonia and Egypt, Europe and America, are furnishing something like a diary of this more modern period. Forgotten and dead languages are beginning to live again. The past is waking up. Our big museums, here and abroad, are flowing over with the treasures of buried ages.

What has all that to do with the subject of this article?

Merely this: Will the school find it profitable and feasible to be in harmony with this spirit of research and evolutionary

insight? In particular, can a universally important subject like mathematics, the oldest of all the sciences, be made more interesting, more purposeful, more real and vital, by being *consciously* linked with the great causative agencies that called it into being and that led to its wonderful growth?

Fortunately, this question has been answered affirmatively by a host of educational leaders. Nor is it necessary to depend *merely* on the monumental contributions of Cantor or his predecessors. Many others have felt the inspiration of the evolutionary, historical point of view. In England, Branford did conspicuous educational work along this line. France has had men like Tannery; Germany had its Simon and Klein; Italy, its Boncompagni and Loria.

It is gratifying that America has had an honorable share in creating this new emphasis on historical mathematical studies. Thus, Hilprecht's volume on the Nippur tablets (1906) has been called "the greatest discovery of all time. relating to Babylonian mathematics," comparable to the translation of the famous Ahmes papyrus of Egypt. *Rara Arithmetica* (1908), by Professor D. E. Smith, threw new light on familiar arithmetical processes. Smith and Karpinski's *Hindu-Arabic Numerals* (1911) is the most authoritative account written on the subject. That little gem, *Number Stories of Long Ago* (1919), also by Professor Smith, urgently calls for a sequel in geometry and algebra. Fine's *Number-System of Algebra* (1890) included an extensive historical sketch. Karpinski's translation of the algebra of Al-Khowarizmi (based on the Latin translation due to Robert of Chester) appeared in 1915. Stamper's *A History of the Teaching of Elementary Geometry* was published in 1909 (Columbia University). It offered a fine bibliography. The National Geometry Syllabus (1912) contained an history introduction prepared by Florian Cajori.

In fact, a very respectable literature on the history of mathematics and its pedagogic significance is now available. Even a brief review of the most important titles would fill many pages. As early as 1874, Hankel insisted on considering the development of mathematics as an integral part of the history of civilization. Gradually, school books began to include historical material. A pioneer effort was made in the excellent text of Balt-

zer, now out of print (sixth edition, 1883). Poul La Cour
wrote a Danish text (1898) with a complete historical setting.
In 1904, at a convention of Swiss secondary teachers, a resolu-
tion was passed demanding that greater attention be given to
historical considerations. Paul Tannery's *Notions Historiques*
appeared in 1903. During the same year, Tropfke's valuable
topical history of elementary mathematics was completed.
Branford's *A Study of Mathematical Education* was issued in
1908, being hailed by Nunn as "the most important and original
of recent English contributions to the pedagogy of mathematics."
Its point of view was entirely evolutionary. A few months
later, Gebhardt's memorable "program" appeared, with its
valuable summary and its excellent new suggestions. The In-
ternational Commission on the Teaching of Mathematics (Rome,
1908) helped very distinctly to stimulate interest in historical
studies. The reports of its numerous sub-committees have
served, up to the present moment, to stress basic questions so
often ignored by the schools.

More recently, the prehistoric and primitive phases of all the
arts and sciences have received much attention. Those who are
interested in "first beginnings" will find the new material ac-
cumulated by the scientific workers in ethnology, anthropology,
and comparative sociology, both fascinating and illuminating.
Such institutions as the American Museum of Natural History
(New York), the Field Museum (Chicago), the Museum of the
University of Pennsylvania, and the United States National
Museum (Washington), have amassed treasures that we are only
beginning to appreciate. (See, for example, *Source Book in
Anthropology*, by A. L. Kroeber and T. T. Waterman, Uni-
versity of California Press, 1920.)

"That is all very well," the inexperienced teacher may say,
"but before I can do anything in this direction, I wish to hear
a more definite statement of the precise purpose of this his-
torical material, and I must also have some idea as to the best
method of using it."

The following paragraphs will attempt to answer these natural
queries very briefly.

Let us consider, first, the cultural and practical aims of his-
torical studies. They may be included under these headings:

1. *Comprehension.*  Without the history of mathematics it is impossible to understand the relative significance and the organic interrelations of its concepts, principles, propositions, and applications.

2. *Appreciation.*  Only through the history of mathematics can we fully realize the unique relation of mathematics to the origin and growth of pure and applied science and to the whole scheme of human existence.  Said Napoleon to Laplace (1812):  ''The welfare of nations is closely connected with progress in mathematics.''   (L'avancement et la perfection des mathématiques sont intimement liés a la prospérité de l'Etat.)

3. *Economy.*  The history of mathematics enables us to avoid the errors of the past and to select methods which are economical and fruitful.

4. *Perspective.*  Rich historical backgrounds keep us from falling an easy prey to the popular slogans of the hour.  A person possessing historical perspective will avoid extremes.  Progress is seen to be due to the constant interaction of theory and practice.  Because Egypt and Rome had a passion for merely *practical* efficiency, and excluded speculation, they produced very few intellectual leaders and their civilization soon became stagnant.  Purely ''theoretical'' studies very often yielded instruments of incomparable value (*e. g.*, conic sections—astronomy, engineering; fractional and negative exponents—logarithms; complex numbers—electrical theory).

5. *Patience.*  The ''golden period of Greek mathematics'' knew nothing of ''minimum essentials'' and ''consumer's needs.''  The ''three great problems'' have a history of many centuries.  Archimedes did not shout ''Eureka'' after a brief ''exposure'' to ''first year mathematics.''  The Greeks knew that there is no royal road to learning except real *work*.  John Napier gave most of his life to the problem of simplifying arithmetical operations.  Fermat's Theorem is not proved to this day.  The temple of mathematics was reared by many generations.  Those who would enter it must be prepared to tarry in its halls and to read its countless tablets with patience and singleness of purpose.

Finally, a few suggestions may be added as to methods.  The writer has experimented in this field of instruction for years.

At the beginning of the course, development lessons and brief talks or illustrated lectures (lantern slides) are desirable. These must be followed later by occasional remarks and references. In the junior high school, the first ten lessons in geometry and algebra are largely historical in character. They are essentially of the form presented in the following pages. Pupils always enjoy this work. It is "different," and by creating a new interest it gives a good start to the school year. The introductory lesson, on the origin of mathematics, has been tried out by the writer in numerous classes (in Rochester, Cleveland, The Lincoln School of Teachers' College, Washington). It has also been frequently repeated by his colleagues. (The outline which is given here was used in the seventh grade. It is based on the writer's personal notes and on reports of observing teachers. Historical details were omitted in this printed summary, for obvious reasons.) All teachers who have experience in this work will render a distinct service by sending their comments or suggestions to the editor of this journal.

### Lesson I. The Origin of Mathematics
#### Time, 40 Minutes

*Purpose of Lesson.* To develop the topic, *The Origin of Mathematics,* and to suggest the great importance of mathematics.

*Plan.* Age of Subject. Hence its necessary character. Situations causing quantitative questions. Their natural character. Necessity of counting. Situations demanding *measurement.* Natural origin of *arithmetic* and *geometry,* and hence of mathematics.

### Outline of Lesson
$(T =$ Teacher)
$(P =$ Pupil)

*Part 1. Age of Mathematics. Its Necessary Character*

*T.* What subject are we about to study?

*P.* Mathematics.

*T.* When you are asked to study any new subject, what questions concerning that subject naturally come to your mind?

$P_1$. We wish to know whether it is worth while.

$P_2$. We want to know "what it is all about."

$P_3$. We wonder how long it will take to master it.

$P_4$. We wish to know how it started.

$T$. Yes, every thoughtful pupil should ask these questions. I cannot answer all of them at once. But I hope that they will all be answered as you go on in your work. Today I shall take up but one of these questions, namely, how mathematics started. You will then begin to see why it is so important. What topic, therefore, are we to discuss today?

$P_1$. The beginning of mathematics.

$P_2$. The age of mathematics.

$P_3$. How mathematics originated.

$T$. In what books do we find information about the origin, or beginning, of our institutions, our government, our customs?

$P$. In historical books.

$T$. There are very interesting books on the history of mathematics. If you were able to read them, you would soon learn that mathematics is the oldest science invented by man. In fact, it is thousands of years old. It is so old that no one can tell just when it started. And the history books tell you that in every country we know about, even thousands of years ago, the people had some knowledge of mathematics. (A brief reference to Babylonia, to Egypt, and to primitive tribes, is desirable at this point.) If that were the only fact you knew about mathematics, what would it indicate to you?

$P$. It is a very necessary (important, essential) subject.

$T$. Yes, and we shall try to see what makes it so necessary. Are people in the habit of keeping a useless thing for thousands of years?

$P$. No, mathematics must have been very necessary at all times.

$T$. Now, if mathematics is such an old subject, and if we find it among all races at all times, what thought does that bring up?

$P_1$. We wonder how it started.

$P_2$. We wish to know what made it so necessary.

$T$. Exactly. Let us think about these questions: How mathematics began, and what made it so necessary. In the first place, do you suppose that some wise man invented it?

$P$. No, because so many people always used it.

## PART II
### Natural Origin of Mathematics.  Shortage Situations.
### Quantitative Questions.

*T.* I shall try to make you see how *natural* it was for people to invent mathematics.  It will not be hard to show you this. Last year I was in New York.  One day we read that all freight trains had stopped running.  When we heard that, what was our impression?

*P.* You thought of how you were going to get food.

*T.* Yes, and also coal, and many other things.  People cannot let such a situation go on without trying to remedy it.  So Mayor Hylan appointed a commission to look into the food situation.  What did they do?

*P.* They found out about the food.

*T.* Yes, and first they went to the storehouses.  What did they ask there?

*P.* "*How much* food have you on hand?"

*T.* What was the next thing to be found out?

*P.* *How many* people were in the city.

*T.* How did they find out?

*P.* They had to count them, or look up the census report.

*T.* How did the managers of storehouses find out the amount of food they had?

*P.* They counted the supplies on hand.

*T.* After they found out how much food they had on hand, and how many people there were, what were they able to tell?

*P.* *How long* the food would last.

*T.* What were the three questions that had to be answered?

*P.* *How much* food have we? *How many* people are in the city? *How long* will the food last?

*T.* How were the answers to these questions found?

*P.* By counting (by "figuring," etc.).

*T.* Now you will probably say that we do not *ordinarily* ask these questions.  Why not?

*P.* We usually have "plenty of food."

*T.* Under what circumstances, therefore, should we *not* ask these questions?

*P.* When there is no shortage.

*T.* Why do we not ask, "How much air have we?"

*P.* We know we have "plenty."

*T.* But that is not always the case. Last year the newspapers reported a situation which *did* make it necessary to ask, "How much air have we?" Do you recall that situation?

*P.* A submarine became stuck in the mud under water.

*T.* Why did the sailors worry about the air in that case?

*P.* They could not live without air.

*T.* We certainly must have food and air. But there are many other situations similar to the ones we have talked about. When do we ask the questions you have mentioned?

*P.* When there is a shortage of something that we must have in order *to live.* (Refer to starving children of Europe at this point.)

*T.* What other shortage except a food shortage have you heard of?

*P.* A shortage of houses.

*T.* Yes, New York needs 100,000 new apartments, and many new houses are needed *here.* How did they find it out?

*P.* By counting ("estimating," etc.).

*T.* How can the house shortage be remedied?

*P.* By building new houses.

*T.* Is building as cheap as it used to be?

*P.* No, all the prices have gone up.

*T.* Can you give the reason for this?

*P.* Building material and labor cost so much more.

*T.* Yes. But why is building material, such as lumber, so much more expensive?

*P.* There is a shortage of lumber.

*T.* That is true. Let us consider that situation for a moment. Years ago this country was covered with forests. There was no shortage of lumber then. In fact, we are told that farmers used black walnut rails for fences. Today we know almost exactly how many walnut trees remain in any locality. Why didn't they count them formerly?

*P.* They had a great many trees.

*T.* Repeat the questions that arise naturally when there is a shortage.

*P. How much* have we? *How many* people must be taken care of? *How long* will the supplies last?

## Part III
### Counting and Measuring

*T.* Notice all these questions again. What causes people to ask these and similar questions?

*P.* Shortage situations.

*T.* Now you will readily understand that as soon as these questions:

> How much?
>
> How many?
>
> How long?

were being asked, people had to think of some way of answering them. But before we go on, tell me whether we could possibly avoid asking these questions *today?*

*P.* (All agree that many everyday situations bring up these questions. Imagine a child going to school in the morning. "*How much* time have I to get to school? *How far* is it to the school? *How much* money must I take along?" Father and mother ask similar questions many times a day. "*How much* does sugar cost? *How many* pounds can I get for one dollar?")

*T.* (Again calls attention to necessary and natural character of these questions, and hence emphasizes the natural origin of mathematics.)

*T.* Of course, all these questions have to be answered. What is the only way of answering a question beginning with "*how many*"?

*P.* (One pupil, a girl, suggests "*so* many." Another child again refers to *counting.*)

*T.* Remember that people could not count at first the way *we* do. But you can all see that they *had* to count. There is no other way of finding out the quantity of supplies you have on hand. We shall learn, by and by, how people counted. Tell me now what, in your opinion, was the final outcome of all this counting.

*P.* (After some hesitation, arithmetic is mentioned.)

*T.* Yes, counting produced arithmetic. And now recall for a moment what you said about the shortage of houses, that is so distressing just now. Of course, you all know why we are living in houses, and why such a shortage is so serious. You may tell me about it.

*P.* We must have houses for protection against heat and cold and wild animals.

*T.* So we may say that houses, as well as food and water and air, are really necessary if we wish *to live.* (Attention is then called to the fundamental human needs, with special emphasis on food, clothing, and shelter.) Suppose now that you are going to build a shelter. Building material, as you know, is not always easy to obtain, and prices have been going up for a long time. What question, therefore, becomes very important?

*P.* How large is the house going to be?

*T.* In other words, what item stands out as a very big one?

*P.* The size of the house.

*T.* And how would the size of the house be determined?

*P.* By measurement. (Repeated questions are necessary to secure this answer.)

*T.* At all times there were many occasions for measurement, just as there were many occasions for counting. You will understand, after all this discussion, that both counting and measuring are necessary and natural processes. They are the foundation of all mathematics; and if someone should ask you, now, how mathematics began, what would you say?

*P.* Mathematics began as soon as people had to count and to measure.

*T.* Let us consider one more question. What was the final outcome, in your opinion, of all the measuring that was necessary in so many situations?

*P.* (No pupil can suggest the correct answer. Some say, "arithmetic"; others, "mathematics." Finally, one boy mentions, "geometry.")

*T.* (Writes "geometry" on the blackboard, and analyzes its constituent parts, comparing the word with "geography." Thus, we have

geo—graphy   and

geo—metry.

It is then seen that geometry really means *"earth measurement"* or surveying.)

## Part IV

### Review and Summary

1. Outline: Origin of Mathematics.   (Oral composition.)

    1. Age and extent.
    2. Importance.
    3. Shortage situations.
    4. Questions.   (Food questions; shelter questions.)
    5. Answers.
    6. Counting—Arithmetic.
    7. Measurement—Geometry.
    8. The Foundation of Mathematics.

2. Key Words of Lesson:

    *Age of Mathematics.*
    *Necessary*—shortage situations.
    *Natural*—food, shelter, clothing.
    *Counting*—arithmetic.
    *Measuring*—geometry.

3. Written Summary:

### *The Origin of Mathematics*

Mathematics is the oldest of all the sciences.   This shows that it must have been very necessary.   It began whenever people asked such questions as, how much? how many? how long? These question were asked whenever there was a shortage or a scarcity of things which we must have.   The only way in which these questions could be answered was by counting and measuring.   Counting led to arithmetic, and measuring led to geometry.   Arithmetic and geometry are the foundation of mathematics.

4. Assignment:   Pupils are to write a brief account of (1) the origin of mathematics, (2) the value of mathematics as demonstrated by the situation which would arise if all mathematics were suddenly destroyed.   They are to find in news papers reports of shortage situations and bring these reports to school.

# ROBERT RECORDE

By Professor FLORIAN CAJORI
University of California

Robert Recorde is the morning star of English mathematical literature, the first writer of note to use the English language as the vehicle of thought in arithmetic, algebra and geometry. Nor were his intellectual interests confined to mathematics; he wrote on medicine and, it is said, was physician to Edward VI and Queen Mary. He and the astrologer John Dee were among the very earliest in England to adopt the Copernican system. In the Latin language mathematical texts of note had appeared somewhat earlier in England. Bishop Tonstall's classic work on arithmetic, the *De Arte Supputandi*, bears the date of 1522; Recorde's arithmetic, *The Grovnd of Artes*, appeared about twenty years later. As regards geometry, De Morgan states that Bishop Robert Greathead's *Compendium Sphaerae* was published in 1531. Recorde's geometry, entitled *The Pathway to Knowledg*, was printed in 1551 and antedated Billingsley's famous English edition of Euclid's *Elements* by nineteen years. But Recorde's algebra, *The Whetstone of Witte*, has no forerunners in England in any language.

Of the three books of Recorde which we have mentioned, *The Grovnd of Artes* enjoyed by far the widest circulation. The date of the first edition is uncertain, but falls somewhere in the interval 1540 to 1543. The oldest edition in the British Museum bears the date 1543; that copy gives no evidence of the existence of an earlier edition. If *The Groud of Artes* (as the title was printed in 1543) did appear some time in 1540-1542, then it is surprising that a new edition should be called for as early as 1543. The edition next following that of 1543 is dated 1549. As long as no copy of an edition antedating 1543 can be found anywhere, the probability is that this is the earliest. During the sixteenth century there appeared a dozen or more editions. After Recorde's death, the book was augmented by John Dee, later by John Mellis, Robert Norton, Robert Hartwell, Thomas Willsford and Edward Hatton. An edition by Hatton bears the date 1699. The book held its place in schools for over 150 years. Recorde's other books enjoyed no such popularity. His *Pathway to*

*Knowledg* passed through three editions, his *Whetstone of Witte* failed to reach a second edition.

The *Whetstone of Witte* was published in London shortly before the author's death in 1558.* His "Dedicatorie" of this work is dated "At London the .XII. daie of Nouember, 1557." At the end of the book the explanation of "vniuersalle rootes" or the roots of binomials of the form $a + \sqrt{b^2}$ is interrupted by the Master's words:

"But harke, what meaneth that hastie knockyng at the doore: *Scholar.* It is a messenger.

*Master.* What is the message; tel me in mine eare. Yea Sir is that the matter? Then is there noe remedie, but that I must neglect all studies, and teaching, for to withstande those daungers. My fortune is not so good, to haue quiete tyme to teache. . . . I mighte haue been quietly permitted, to reste but a little lōger."

His reference to his "fortune" suggests financial embarrassments; this much is certain, he died the year following in the King's bench prison at Southwark. The usual assertion that he was imprisoned for debt is rendered doubtful by the fact that in his will of June 28, 1558, he left a little money to his mother and brother and to his officers and fellow prisoners. In his will he styles himself, "Robert Recorde, doctor of physicke, though sicke in body yet whole of mynde." Recorde is not the only mathematician said to have been imprisoned for debt. His Italian contemporary, Hieronimo Cardan, likewise a mathematician and physician, shared this fate.

Writing on geometry, algebra and astronomy in the English language necessitated the introduction of English technical terms for those sciences. Recorde's first publication in geometry, his *Pathway to Knowledg*, 1551, is particularly interesting on this point. On a leaflet, dated May 20, 1665, Isaac Newton says he used "pricked letters" to denote fluxions. To us "pricked" suggests "pierced" or "stung," but not to Newton or Recorde. "A Poynt or a Prycke," says Recorde in his *Pathway to Knowledg* "is named of Geometricians that

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* Readers interested in early mathematics in England may consult A. DeMorgan's valuable article, "Notices of English Mathematical and Astronomical Writers between the Norman Conquest and the Year 1600" in the *Companion to the British Almanac* for 1837, pp. 21-44.

small and vnsensible shape, whiche hath in it no partes.''.   A
circle "hath in the middell of it a pricke or centre." Recorde's
Saxon-English "sharpe" and "blunt" angles or "corners" are
no less desirable than our Latin "acute" and "obtuse" angles.
His "paralleles, or gemowe lynes" are "suche lines as be
drawen foorth still in one distaunce, and are no nerer in one
place then in an other." "Gemowe lynes" need not be straight;
they may be "tortuouse paralleles," like the letters SS.   This
notion of "tortuouse paralleles" led Recorde to make the follow-
ing criticism : "Here might I note the error of good Albert Dur-
er, which affirmeth that no perpendicular lines can be paral-
leles." True to his Anglo-Saxon, Recorde calls a tangent line
a "touche lyne," vertical angles "matche corners." An equi-
lateral triangle he calls a "threlike," an isosceles one, "twey-
like." Then follow "quadrangles, which are figures of iiij.
corners and of iiij. lines also." His "square quadrate" is what
we now term a "square;" his "long square" is our rectangle, his
"losenges or diamondes" are our "rhombs," his "losengelike"
or "diamōdlike" is our "oblique parallelogram." Parallelo-
grams which are either right or oblique, are designated by the
oddly sounding name of "likejammys." One of Recorde's con-
structions is "to make a likeiāme equall to a triangle appointed,
and that in a right lined āgle limited" (i. e., to construct a par-
allelogram equal to a given triangle and having a prescribed
angle). Trapeziums are called "borde formes, they have no syde
equall to an other."

Pentagons,, hexagons and heptagons are called "cinkangles,"
"siseangles" and "septangles." Certainly a "plumme line" is
preferable to our "perpendicular line."

One reason why Recorde's terminology did not find adoption,
may be inferred from the fact that for over a century after his
day, the language of scholars continued to be Latin and the more
important and influential mathematical works were mostly in
Latin.  Of William Oughtred's *Clavis Mathematicae*, which ap-
peared during the 17th century in seven editions, five were in
Latin and only two in English.  John Wallis' *Algebra* of 1685,
was in 1693 translated by him from English into Latin.  With
Latin as the dominating language, there was little opportunity

for Saxon-English technical phraseology of the sixteenth century to become fixed.

Probably there were other reasons why Recorde's terminology in geometry failed to establish itself permanently. Some of his words carry an unpleasant sound to us and may have done so to his countrymen. At any rate, nineteen years after the issue of the *Pathway to Knowledg,* Billingsley uses in his Euclid (1570) the term "long square" for rectangle, "Diamond figure," or "diamonde" as an alternate for rhombus, but otherwise he makes no use of the terms given by Recorde; Billingsley does not speak of "prickes," "tweylike triangles," "gemowe lynes," and "likejammys."

A distant approach to our linear continuum is found in the following Recordian sentences: "Nowe of a great numbre of these prickes, is made a Lyne, as you may perceiue by this forme ensuying. . . . . . . . where as I have set a numbre of prickes, so if you with your pen will set in more other prickes betweene euerye two of these, then wil it be a lyne, as here you may see —— and this *lyne* is called of Geometricians, *Lengthe withoute breath.*" This exposition is referred to as one of "theyr theorikes (which ar only mind workes)."

In *The Pathway to Knowledg* Recorde gave, besides definitions, 46 constructions, called "conclusyons," Euclid's postulates and axioms ("certaine grauntable requestes" and "certayn common sentences manifest to sence") and finally 77 theorems: "I go on with the theoremes, whiche I do only by examples declae, minding to reserue the proofes to a peculiar boke which I will then set forth, when I perceaue this to be thankfully taken of the readers of it."

Of historical interest are some of the technical terms in Recorde's *Whetstone of Witte,* for they indicate the migration of algebra from country to country. The name "Cossike nombers" for numbers represented by algebraic signs is evidently borrowed from German authors (Christoff Rudolff, Michael Stifel, Johann Scheubel and others) who spoke of "Cossische Zahlen" and sometimes called Algebra, "Die Coss." This German terminology in its turn is an adaptation from the Italian word "cosa" (thing) which was used by the Monk Luca Pacioli (1494) and by others, to designate the first power of an unknown

quantity. As the German 'Coss'' and the Italian "Cosa" sound much like the Latin word "Cos" (whetstone), it was conjectured by A. DeMorgan* that Recorde's *Whetstone of Witte* was a punning title. It may seem ungracious to spoil an interpretation so ingenious but our recent study of Recorde has led us to the conclusion that Recorde did not cultivate the fine art of punning but indulged in less subtle literary forms, revelling in such pictorial language for the title-pages of his books as *Castle of Knowledg* (astronomy), *Grovnd of Artes* (arithmetic), *Pathway to Knowledg* (geometry). The phrase "whette stone of witte" was not freshly coined by him especially for his book on algebra; it was used by him in a more general sense on earlier occasions. In the *Epistle to the Kings M. A.*, printed in his *Pathway to Knowledg,* he speaks of "wyttie men" who "by the readying of wyttie artes (whiche be as the whette stones of witte)" may "increase more and more in wysedome." There is no particular and exclusive reference here to algebra or "Coss." Apparently, when he wrote his algebra, he seized upon one of his habitual phrases, the "whetstone of witte" as a fitting title for that book.

Another word which found its way from Italy to Germany and thence to England is "cento," used by Pacioli and others to indicate the square of an unknown number, *i. e.*, $x^2$. For the purpose of imitating the Italian sound of the word, the Germans changed the "c" to "z." Some German authors used the latinized form "zensus" to designate $x^2$, and adopted the abbreviation "z." Thus, 15.$z$ stood for $15x^2$; 10.$zz$ for $10x^4$; moreover, $\sqrt{zz}$ stood for "fourth root." This symbolism passed to England and was employed by Recorde. Hence we see that the Italian "cento" became the German "zensus" and the English "Zenzike." Says Recorde: "You shal vnderstande, that many men doe euer more call square nombers *zenzikes,* as a shorter and apter name." Similarly, Recorde writes "zenzizenzike" for the fourth power, "zenzicubike" for the sixth, "zenzizenzizenzike" for the eighth, "zenzizenzicubike" for the twelfth.

He was the first Englishman to use the German symbols $+$ and $-$. As is well known, Recorde invented our sign for equality.

* A. DeMorgan. Arithmetical Books. London. 1847, p. 21.

"I will sette as I doe often in woorke vse, a paire of paralleles, or Gemowe lines of one lengthe, thus: ========, bicause noe. 2. thynges , can be moare equalle." It is interesting to observe that this sign of equality is the only new symbol suggested by Recorde, but that one was so well selected, that mathematicians have been unwilling to let it die. In fact, no less influential a writer than Descartes endeavored to introduce another sign, but failed in his attempt. The history of mathematics shows many algebraists who suggested new symbols but failed to have any of them permanently incorporated into the science. Recorde's success in this matter was unique. In view of the fact that Recorde in all sincerity explains the origin of the sign of equality, it seems bad judgment for Charles Henry* to endeavor to establish its origin in the parallel strokes, sometimes found in medieval manuscripts, to represent *id est*.

DeMorgan has pointed out that in the *Castle of Knowledg* Recorde mentions the "arte of *sines* and cordes" which is the first time we have found the word "sine" in English.

Recorde cannot be said to stand out in the history of mathematics as a conspicuously creative mind. No great contribution on a level with the Italian solution of cubic equations or the Dutch invention of decimal fractions or the Vietian method of approximation to the roots of numerical equations or the clearer interpretation of negative and imaginary numbers or the recognition of new theorems in the theory of equations, stands to his credit. Only minor entries can be made on the credit side of his account, such as a simplified presentation of the solution of linear and quadratic equations. On cubic equations he says nothing. In most respects Recorde's exposition is along traditional lines. Perhaps the most signal example of this is his nomenclature in connection with ratios. The Greek Nicomachus, probably known to Recorde through the arithmetic of Boethius, had shown in his arithmetic an extraordinary predilection for complicated words to designate simple ratios, such as 3/2, 4/3, 2/3, etc. In imitation of this practice, Recorde calls the ratios 3/2 sesquialtera, 4/3 sesquitertia, 5/3 superbipartiens tertia, 3/4 suptripartiens quarta. Some seven pages are devoted to such names, first by recording the Latin names and next by discussing certain English equivalents.

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* C. Henry in *Revue Archéologique*, N. S., Vol. 38, 1879, p. 5.

Recorde's texts are written in the form of a dialogue between Master and Scholar. He states his reason for doing so in his preface to the *Grovnd of Artes*: "because I judge that to be the easiest way of instruction, when the Scholar may ask every doubt orderly, and the Master may answer to his questions plainly." Recorde indeed anticipates many of the difficulties a pupil will encounter. Frequently quoted is the following passage in the *Grovnd of Artes*, which illustrates the occurence of rhyming sentences in the body of the text, such as are seldom found in his later books:

"*Scholar.* And I to your authority, my wit doe subdue, whatsoever you say, I take it for true.

"*Master.* That is too much, and meet for no man to be believed in all things, without shewing of reason. Though I might of my Scholar some credence require, yet except I shew reason, I doe it not desire."

Reading his text, the modern reader infers that Recorde must have been an excellent teacher, as indeed he was, if we may trust his biographers. In his *Whetstone of Witte*, he solves a number of algebraic problems, of which the first was as follows:

"*Master.* Take this for the firste question. Alexander beyng asked how olde he was, I am .2. yeres elder (quod he) then Epheuw. Yea, saied Epheuw. And my father was as olde as we bothe, and .4. yeres moare. And my father hawyng all those yeres, saied Alexander, was .96. yeres of age. I demaunde now of you, how olde was eche of them."

To this the Scholar replies humorously, "I praie you aunswere the question yourself, to teach me the forme."

The Master explains, then proceeds to a problem involving a question of debt, and remarks: "There is nothyng better then exercise, in attainyng any kynde of knowledg." Then follow questions on the size of armies, the heights of walls, the number of "bricke," the division of an estate, etc. In one place the Scholar remarks, "Although the question seme harde, I see many tymes, that diligence maketh harde thynges easie, and therefore I will attempte the worke of it."

With Recorde a negative number is "absurde" and "expresseth lesse than naught"; nevertheless there are passages which indicate that he did not reject them. "Howbeit for ex-

amples sake, you maie woorke, as well with *Absurde* nombers, as
with any other.''

Great interest is shown at the present time in the history
of science during the middle ages.   Roger Bacon, in particular,
is being studied with renewed zeal.   It is worth while, therefore,
to quote Recorde's remark touching the long mooted question of
Bacon's invention of the telescope.   In the preface to the *Path-
way to Knowledg* he says:

''Great talke there is of a glasse that he made in Oxforde, in
whiche men myght see thynges that were doon in other places,
and that was iudged to be done by power or euyll spirites.   But
I knowe the reason of it to bee good and naturall, and to be
wrought by geometrie (sythe perspectiue is a parte of it) and to
stande as well with reason as to see your face in cõmon glasse.''

And indeed ''great talke there is'' at the very present time,
when the daily press brings news of a Baconian book written in
cypher that is now being deciphered at the University of Penn-
sylvania and which appears to reveal that Roger Bacon in the
thirteenth century really possessed a telescope and microscope.
Our histories of physics assign the invention of these instru-
ments to the beginning of the sevententh century, but Recorde
made the above remark at the middle of the sixteenth century.

Recorde referred to the telescope when he was indicating the
the applicability of geometry to science.   He was convinced not
only of the practical usefulness of arithmetic, algebra and geo-
metry, but also of their value in training the mind.   He quotes
from Plato's *Republic* on these points.   His convictions in this
matter are brought out on the very title page of the *Whetstone
of Witte,* which in closing we copy in full:

The whetstone
of witte,
whiche is the seconde parte of
Arithmetike: containyng thextrac-
tion of Rootes:   The *Cossike* practise,
with the rule of *Equation*: and
the woorkes of *Surde
Nombers.*
Though many stones doe beare greate price,
The *whetstone* is for exersice.

As neadefull, and in woorke as straunge:
Dulle thinges and harde it will so chaunge,
And make them sharpe, to right good vse:
All artesmen knowe, thei can not chuse,
But vse his helpe: yet as men see,
Noe sharpenesse semeth in it to bee.

  The *grounde of artes* did brede this stone:
His vse is greate, and moare then one.
Here if you list your wittes to whette,
Moche sharpenesse therby shall you gette.
Dulle wittes hereby doe greately mende,
Sharpe wittes are fined to their fulle ende.
Now proue, and praise, as you doe finde,
And to your self be not vnkinde.

  These Bookes are to bee solde, at
  the Weste doore of Poules,
  by Jhon Kyngstone.

# A LIST OF REFERENCE BOOKS AND MAGAZINES
## FOR TEACHERS OF MATHEMATICS

By W. D. REEVE
University of Minnesota

I. Books of a general pedagogical nature:

1. Judd, C. H., *The Psychology of High School Subjects*. Ginn and Co.

2. Parker, S. C., *Methods of Teaching in High Schools*. Ginn and Co.

These two books are the best of their kind available and are extremely interesting and helpful. Every teacher of high school mathematics should read them.

II. Books on the teaching of mathematics:

1. Branford, B., *A Study of Mathematical Education*. Clarendon Press. This is an exceedingly valuable book for both the elementary and the high school teacher.

2. Carson, G. St. L., *Mathematical Education*. Ginn and Co.

3. Evans, G. W., *The Teaching of High School Mathematics*. Houghton Mifflin Co.

4. Nunn, T. P., *The Teaching of Algebra, Including Trigonometry*. Longmans, Green and Co.

This book of Nunn's has two companion volumes which are very helpful. They are *Exercises in Algebra, Including Trigonometry*, Part I and Part II.

5. Perry, John, *Report of the British Association at Glasgow*, 1901. The Macmillan Co., 1901. (Out of print.)

6. Safford, T. H., *Mathematical Teaching*. D. C. Heath and Co.

7. Schultze, A., *The Teaching of Mathematics in Secondary Schools*. The Macmillan Co.

8. Smith, D. E., *The Teaching of Elementary Mathematics*. The Macmillan Co.

9. Smith, D. E., *The Teaching of Geometry*. Ginn and Co.

10. Young, J. W. A., *The Teaching of Mathematics*. Long-

11. Young, J. W. A., *The Teaching of Mathematics in Prussia*. Longmans, Green and Co.

III.   Books relating to mathematical topics intended to improve
the teacher of mathematics.

1. Chrystal, George, *Algebra*, 2 vols.   A. and C. Black.

2. Conant, A., *The Number Concept*.   The Macmillan Co.

3. De Morgan, Augustus, *Study and Difficulties of Mathe
matics*.   Open Court Publishing Co., 1910.

4. Fine, H. B., *College Algebra*.   Ginn and Co.

5. Fine, H. B., *The Number System of Algebra*.   D. C. Heath
and Co., 1901.

6. Heath, Thomas, *The Thirteen Books of Euclid*.   Cambridge
Press.

7. Hilbert, David, *The Foundations of Geometry*. Open Court
Publishing Co., 1910.

8. Keyser, Cassius J., *The Human Worth of Rigorous Think-
ing*.   Columbia University Press.

9. Klein, Felix, *Famous Problems in Elementary Geometry*.
Ginn and Co.

10. Lagrange, Joseph, *Lectures on Elementary Mathematics*.
Open Court Publishing Co.

11. Laisant, C. A., *Initiation Mathematique*.   This book is in
French but is easy reading and has a refreshing outlook on
mathematics.   It has a subtitle "A Work Entirely Disconnected
with Any Program and Dedicated to the Friends of Children."

12. Manning, H. P., *Non-Euclidean Geometry*.   Ginn and Co.

13. Manning, H. P., *Geometry of Four Dimensions*.   The
Macmillan Co.

14. Manning, H. P., *Fourth Dimension Simply Explained*.
Munn and Co.

15. Manning, H. P., *Irrational Numbers and Their Repre-
sentation by Sequences and Series*.   Wiley and Sons.

16. Poincaire, Henri, *Science and Hypothesis*.   Science Press.

17. Russell, Bertrand, *Essay on the Foundations of Geometry*.
Cambridge Press.   (Out of print.)

18. Slosson, Edward, *Easy Lessons in Einstein*.   Harcourt.
Brace and Co.

19. Smith, D. E., and Karpinski, *The Hindu-Arabic Numerals*.

20. Whithead, A. N., *Introduction to Mathematics*.   Henry
Holt and Co.

21. Withers, J. W., *Euclid's Parallel Postulate*. Open Court Publishing Co.

22. Young, J. W., *Fundamental Concepts of Algebra and Geometry*. Macmillan and Co.

23. Young, J. W. A., *Monographs on Topics and Elementary Mathematics Relevant to the Elementary Field*.

IV. Books containing recreational material.

1. Abbot, *Flatland*. Little, Brown and Co. (Out of print.)

2. Ball, W. R., *Mathematical Recreations and Problems*. Macmillan Co.

3. Cheever, E. J., *The King of Calculators*. E. J. Cheever, Little Rock, Ark.

4. De Morgan, Augustus, *Budget of Paradoxes*. Open Court Pub. Co.

5. Jones, S. I., *Mathematical Wrinkles*. S. I. Jones, Nashville.

6. Leacock, Stephen, *Literary Lapses*. John Lane, N. Y.

7. Licks, H. E., *Recreations in Mathematics*. D. Van Nostrand Co., N. Y.

8. Manning, H. P., *Fourth Dimension*. Munn and Co.

9. Perry, John, *Spinning Tops*.

10. Schubert, Hermann, *Mathematical Essays*. Open Court Pub. Co.

11. Sykes, Mabel, *Source Book for Geometry*. Allyn and Bacon.

12. Thompson, Sylvanus, *Calculus Made Easy*. Macmillan Co.

13. White, W. F., *A Scrap-Book of Elementary Mathematics*. Open Court Publishing Co., 1910.

V. Books of a Historical Nature.

1. Allman, George J., *Greek Geometry from Thales to Euclid*. Longmans, Green and Co.
Macmillan Co.

3. Cajori, Florian, *A History of Mathematics*. The Macmillan Co.

4. Ball, W. R., *A Short History of Mathematics*. The Macmillan Co.

5. Fink, Karl, *A Brief History of Mathematics*. Open Court Pub. Co.

6. Gow, James, *A Short History of Greek Mathematics.* Cambridge Press.

7. Jacobs, Hermann, *The Seven-Tellers.* Hermann Von Jacobs, Berlin.

8. Mach, Ernst, *Science of Mechanics.* Open Court Publishing Co. (Reads like a novel and will keep the teacher in wide-awake touch with the mathematics work as it connects with physics.)

9. Stamper, A. W., *A History of the Teaching of Elementary Geometry.* Teachers' College, Columbia University, N. Y.

VI. Books Available as Texts.

A representative selection from all of the standard older texts and the modern texts on mathematics should be in the teacher's library and he should be able to choose from this list texts that would suit his purpose.

For a complete list of such books, see the Report of the Committee of Bibliography, p. 31, THE MATHEMATICS TEACHER, September, 1915, Vol. 8, No. 1.

VII. Magazines.

1. Of a Mathematical Nature.

*a. The Mathematics Teacher*, 71 South Broadway, Yonkers, N. Y.

This is the official organ of the National Council of Mathematics Teachers and is the only journal dealing entirely with mathematics in the secondary school. It contains material that will be helpful to mathematics teachers and every mathematics teacher should take it.

*b. The American Mathematical Monthly.* Treasurer, W. D. Cairns, Oberlin, Ohio.

This magazine deals chiefly with mathematics foreign to the secondary school field, but it is a good thing for the teacher below the college and university to make it the means of helping him in an attempt to do advanced work.

*c. School Science and Mathematics.* Mount Morris, Ill.

Though not of a strictly mathematical nature, this journal which is the official organ of the Central Association of Science and Mathematics Teachers contains some very stimulating articles for high school teachers of mathematics and it keeps him in closer touch with the allied sciences.

*d.* Technical magazines.

1. *Scientific American.* Munn and Co., N. Y.

2. *Scientific American Supplement.* Munn and Co.

*e.* Every teacher who can read either French or German should try to read at least one journal in either of these two languages.

2. Of a General Nature.

*a. School Review.* University of Chicago Press.

*b. School and Society.* Science Press, Garrison, N. Y.

These two magazines are two of the most stimulating and helpful sources of information for high school teachers.

## DISCUSSION

*A Few Lessons in Calculus for High Schools.* The National Committee on Mathematical Requirements has stated that some work in calculus could well be taken in high school. As the idea appealed to us, we decided to have such a course here—a very short course, about twenty lessons.

The next thing was to find a suitable textbook, and here arose a serious difficulty. Almost all the American textbooks were written from a college point of view. They were too heavy. Certain English books came much nearer the desired field. As none of these was entirely satisfactory, we decided to arrange our own course.

We made the aim to give the pupil an acquaintance with a few of the simplest applications of calculus. For content we limited the field to the so-called practical problems, leaving out the more abstract work. And as differentiation and integration have many forms we omitted all work with logarithms and trigonometric functions of angles. The three main topics selected for problems were, Variable Rates of Motion, Maxima and Minima, and Finding Areas and Volumes. About twenty-five easy problems were collected for each group, samples of which are given later. Then enough theory with exercises was supplied to prepare for the problems. The drill exercises to a large extent were taken from the problem work. Differentials included the coefficient, the constant term, the sum, the product,

the power, the root, and the quotient. Integrals included the constant term, the coefficient, the sum, and the power. There was much graphic representation. The eight chapters are:

1. First Principles.
2. Differentials.
3. The Derivative.
4. Rates of Motion.

6. The Second Derivative.
7. Integrals.
8. Distance, Area and Volume.
5. Maxima and Minima.

Sample problems follow:

A lamp is 60 ft. above the ground; a stone is dropped from a point at the same level as the lamp and 20 ft. away from it. Find the rate of the shadow of the stone on the ground after 1 sec.; after falling 30 ft.

A boat 15 mi. south of a lighthouse sails due east at the rate of 10 mi. per hour. At what rate is the boat leaving the lighthouse after 2 hours?

A man is walking 4 ft. per second across a bridge; a second man in a motorboat is going at right angles from the bridge 6 ft. per second. At what rate are the men separating after 5 seconds?

The formula for the distance traveled by a stone thrown upward into the air, with a velocity $V_0$, is $s = V_0 T - \frac{1}{2}gt^2$. Find the greatest height reached by the stone.

A window is in the form of a rectangle surmounted by a semi-circle. If the perimeter is 30 ft., find the width so that the greatest possible amount of light may be admitted.

The number of tons of coal consumed per hour by a ship is $.3 + .001V^3$ where V is the speed. For a voyage of 1,000 mi., find the total consumption of coal; then find for what speed the coal consumption is the least.

Find where the curve $y = 1 - x^2$ cuts the x axis and then find the area between the curve and the x axis.

Find the volume generated by the hyperbola $y = \sqrt{1 + x^2}$ revolved about the x axis, between the limits $x = 0$ and $x = 4$.

Find where the ellipse $4x^2 + 9y^2 = 36$ cuts the y axis and then find the volume of the ellipsoid made by the ellipse revolved about the y axis.

A boat is fastened to a rope which passes over a wheel 20 ft. above the level of the boat. If the rope is passing out at the rate of 8 ft. per second, how far is the boat from the wharf? The boat is drifting down stream at the rate of 10 ft. per second constantly.

This course in mimeographed form has been tried out in an elective class of the better type of seniors meeting once a week and has held their interest and activity.

ROBERT R. GOFF.

Academic High School, New Britain, Conn.

*Vitalizing Instruction in Geometry.* I visited a class in geometry in which all of the girls—it was a class of girls only—went to the blackboard where a carefully drawn copy of the figure "in the book" lettered just so, and carefully set so that it would not tip over, awaited them.

Each girl in her turn went through the demonstration in the
exact words of "the book," and everybody, teacher and pupils,
was happy—except the visitor.

The theorem set up the claim—and proved it—that a line
drawn parallel to the base of a right angled triangle divides
the other two sides proportionally.

Believing that no one of the seven really understood the the-
orem, or appreciated its value, I could not quite console myself
with the sage reflection that fortunately they would forget the
whole thing in a short time.

Therefore, after gaining permission from the teacher, I put
this to the girls:

"Once there were three men—Jones, Smith and Robinson
—who owned town lots fifty feet wide, side by side, front-
ing on Elm Street, like this:
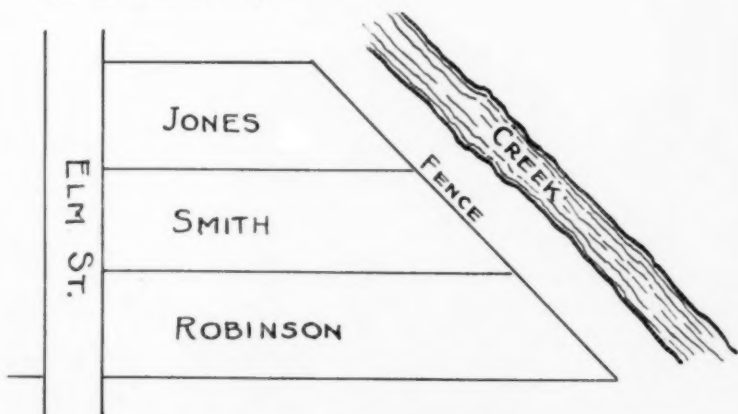


"Back of their lots ran a creek, and since they all had
little children who might fall in and be drowned, they hired
a man to build a wire fence at the rear of their lots along
the creek. When he presented the bill, $28.50, the question
arose how they should divide it. What do you think about
it? Which of the three should pay the largest part of the
bill?"

We took a vote and every girl voted for Robinson. Why? I
never found out. After they were thoroughly committed, I
completed the triangle and showed them that each of them had

just carefully proved to me that the bill should be equally divided—\$9.50 for each man.

There were "Oh's" and "Ah's," and there was genuine surprise that anything in a geometry book had anything to do with real life. Even the teacher showed similar symptoms!

I remembered what some cruel iconoclast had said to the effect that geometry was invented as a means of intellectual recreation for a group of highly trained philosophers in Athens whose minds sought for statements and processes of logic as thoroughly abstract as possible—and then we feed it—just as it has lain in its two thousand year packing—to children fifteen years old. No wonder the more enterprising classes burn their geometries with ceremony!

And I asked myself why we could not have all, or most of our geometrical theorems presented as concrete problems taken from familiar environment and let the pupils guess at it a while —even if they "guess" Robinson!—and then assign "for tomorrow" a theorem which will tell how to divide the bill fairly.

Youthful psychology is a fairly safe guide in lesson assignment, O ye pedagogues, to say nothing of classroom conduct.

And, would it be too bold to suggest that youthful psychology be given some consideration in the writing of textbooks?

Unless we heed this lesson, the reformers are going to sweep out all the broken idols—beautiful marble though they may be— and substitute mechanical puppets and toys, because they have joints and youth can work them!

All reforms should come from *within*. Most of them come from *without*, and are reckless and destructive and wasteful.

<div style="text-align:right">JOHN CALVIN HANNA.</div>

Supervisor of High Schools for Illinois.

## NEWS AND NOTES

PROFESSOR MARTIN NORDGAARD has been appointed head of the department of mathematics in Antioch College. He was recently granted the Ph.D. degree by Columbia University.

MR. RALPH BEATLEY, for some time head of the department of mathematics in the Horace Mann School for Boys, in New York City, has been appointed to an important position in Harvard University. He will have charge of the professional courses in the teaching and supervision of mathematics.

THE Ohio State Educational Conference is stimulating research among the various sections. The program of the Mathematics Section of the second annual meeting (March, 1922) consisted of:

Address, "The Benefits of Grouping Students in Mathematics on the Basis of Ability," D. W. Werremeyer, West Technical High School, Cleveland.

Address, "The Content of the Course in Mathematics for the Junior High School," Amy F. Preston, Roosevelt High School, Columbus.

Address, "Should the Content of Mathematical Courses Be Limited to the Needs of the Child?" J. C. Boldt, Steele High School, Dayton.

M. M. Mansperger, Principal of Barnesville High School, presiding.

TEACHERS of mathematics will be interested in the Junior Mathematical and Drawing Set which the Keuffel & Esser Company (New York and Chicago) has recently announced. It is an inexpensive, high grade set consisting of a pair of compasses, a brass protractor, enameled in a way to be easily read, an inch-metric ruler and samples of cross section paper. The new junior high school courses, as well as the standard introduction to demonstrative geometry, present a very real need for a set like this one. It sells for about thirty-five cents.

THE National Committee on Mathematical Requirements regrets to announce that, owing to unfortunate delays in printing, its complete report, "The Reorganization of Mathematics in Secondary Education," will not be ready for distribution before

May, 1922. Requests for free copies of this 500-page report may be sent to J. W. Young, Chairman, Hanover, New Hampshire. Owing to the labor and expense involved, the receipt of applications for copies of the report is not in general being individually acknowledged. Applicants may rest assured, however, that their requests will be filled when the report is ready for distribution.

THE New York Section of the Association of Teachers of Mathematics in the Middle States and Maryland held its spring meeting on March 10, 1922.

"How to Teach," was the general topic of the program. The specific subjects and speakers were: "How to Teach the First Three Lessons of Geometry," Mr. Charles Burton Walsh; "How to Teach the First Three Lessons of Algebra," Miss Ruth Barry; "Some Things Worth Teaching to a Ninth Grade Class of Boys," Mr. Ralph Beatley; "Some Things Worth Teaching in an Appreciation Course for Girls in the Tenth School Year," Miss Vevia Blair; "How to Teach Graphs," Mrs. Jean F. Brown; "How to Teach an Operative System of Symbols in Metrical Geometry," Mr. Howard F. Hart; "How to Teach Addition and Subtraction of Fractions," Mr. John R. Clark. Raleigh Schorling is chairman of the section.

THE spring meeting of the Association of Teachers of Mathematics in the Middle States and Maryland was held Saturday, May 6, 1922, at Teachers' College, Columbia University.

Professor Edward L. Thorndike discussed "The Psychology of Drill in Algebra."

Dr. Ben D. Wood spoke on "Suggested Improvements in Mathematics Examinations."

The officers of the association are: President, Charles B. Walsh, Friends' Central School, Philadelphia, Pa.; Vice President, Elizabeth Johnson, Baldwin School, Bryn Mawr, Pa.; Secretary, John C. Bechtel, Germantown High School, Philadelphia, Pa.; Treasurer, Clarence P. Scoborio, Polytechnic Preparatory School, Brooklyn, N. Y.

SCIENCE for March 31, 1922, contains various addresses delivered at the recent dedication of the Norman Bridge Labora-

tory of Physics at Pasadena, Cal. This was a notable event in the progress of science. Teachers of mathematics will be interested in the following from the address of Dr. R. A. Millikan delivered on that occasion:

"With the gradual disappearance of the classics and the rigid discipline which they furnished, as the basis of our higher educational system, there have been slowly creeping into it (American education) during the past two decades certain emasculating influences which need to be counteracted. There is no Elisha upon whom the mantle of the classics can fall except the mathematical and physical sciences. There is no training like that which they furnish for teaching men to apply themselves intensively, to observe carefully and correctly, to treat their data honestly and dispassionately, and to reason objectively from a given set of conditions to their inevitable consequences —in a word to see clearly and to think straight. Indeed, there is nothing else left to constitute the backbone of the training of the coming generation if it is to maintain the virility and the strength of those who have preceded. The institute hopes to do some pioneer work in demonstrating the values of an education having the mathematical and physical sciences for its backbone. I accept this gift, then, in behalf of American education in the confident belief that the intensive training in the mathematical and physical sciences which will take place within its walls may exert a wholesome, yes, a saving influence upon American education as a whole.''—By Alfred Davis.

## NEW BOOKS

**Mathematical Philosophy.** A Study of Fate and Freedom. By
  Cassius J. Keyser. E. P. Dutton and Company, New York,
  1922. Pp. 466.

No one who has read Professor Keyser's address on *The Human
Worth of Rigorous Thinking* which appeared in the January
number of the Mathematics Teacher, and which is a part of
the opening chapter of his *Mathematical Philosophy*, can fail to
anticipate that this volume will be keen and stimulating. And
he will not be disappointed.

But one must not be misled by the description "Lectures for
Educated Laymen" which follows the title. This implies a popular
character that the book emphatically lacks, if by popular we
mean a denatured, sugar-coated work so constructed that a person
of normal intelligence can assimilate it with little sensible
effort. On the other hand, Professor Keyser has been so careful
to define the technical terms which he uses that it would be
possible to read the work with appreciation even though, at the
start, one had no acquaintance with such topics as inversion,
invariants, and non-Euclidean geometry. In this respect, the
author has written a book for laymen, but the layman is flattered
by the knowledge that it is for scholars as well. One bit of
caution must be inserted here. Unlike a series of lectures of the
popular variety, these are not units in themselves but are so
interdependent that it is difficult if not impossible to read single
chapters without reading the preceding ones. It is a course
which admits no "cutting."

In the preface, the author states that although the book is
primarily addressed to students in philosophy, he hopes these
lectures "may not be ungrateful to a much wider circle of readers
and scholars" among whom he cites "the growing class of
such teachers of mathematics as endeavor to make the spirit of
their subject dominate its technique." Many of us are members
of this group by self-election. To us therefore, in our conscious
or unconscious search for a better understanding of the
spirit of mathematics, this work makes an especial appeal. We
learn that the Fate and Freedom of the subtitle mean logical
fate embodied in the laws of thought, and intellectual freedom

whose liberty lies in the keeping of these laws. The connection with our subject comes in this quotation:* "Now it so happens that the term mathematics is the name of that discipline which, because it attains more nearly than any other to the level of logical rigor, is better qualified than any other to reveal the prototype of what is best in the quality of thinking as thinking."

It would be sheer presumption for a person who is an amateur in mathematics and not even that in philosophy, to criticise or comment upon the treatment of the subject matter of this volume, and yet I think it may be permissible to say that Professor Keyser gives one a deeper conception of the meaning of terms we use glibly as postulate, function, limit, infinity, and he gives us a glimpse of the soul of mathematics as he himself sees it. To paraphrase the last sentence of his introductory chapter, he discloses the relation of mathematics to other great forms of intellectual activity especially in its bearings upon the universal interests of humanity.

<div align="right">VERA SANDFORD.</div>

The Lincoln School.

**Fundamentals of Practical Mathematics.** By GEORGE WENTWORTH, DAVID EUGENE SMITH, and HERBERT DRUERY HARPER. Ginn and Co. Pp. 202.

In reading this book, the reviewer is impressed with (1) the fact that the book teaches. It is more than a collection of practical problems, for it develops those basic principles which the student must know, whatever vocation he is to follow; (2) the thirty-six full page blue prints, which give a sense of reality to the exercise; and (3) its mechanical and literary excellence. The chapter headings—"Fundamental Operations," "Ratio and Proportion," "Mensuration," "Trigonometry," "The Slide Rule," and "General Applications"—warrant the authors in calling their material the *fundamentals* of practical mathematics, for they have not been bound to academic distinctions between the different divisions or subjects of mathematics.

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*Mathematical Philosophy p. 18.